# The Impact of Bulk Density on Shape of a Cohesive-less Granular Heap

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## Abstract

*We provide an analytical expression for the shape of a heap of cohesive-less granular material with the impact of bulk density. By assuming the bulk density of the heap is changing with the depth of the heap, we prove that the shape of the heap is no longer a perfect conical shape. It is the first time to relate the shape of the heap to the compaction coefficient and porosity of granular matter. We analyze the effects of the different parameters on the shape of the heap. The results are in agreement with the experiment. The model may be applied to extract the quantitative information on the flowability, porosity, and compaction coefficient of a granular material from the shape of its heap.*

**Keywords:** Bulk density, Particle density, Particle technology

## Introduction

A heap constructed by pouring cohesive-less powder from a fixed point forms a conical shape. The angle of θ in Fig.1 is called the angle of repose. Let $\mu = \tan(\theta)$, μ is named the coefficient of internal friction. The angle of repose is an important value to measure the flowability of a powder. Many methods to estimate the angle are based on this ideal conical shape derived from the Coulomb material theory [3, 4]. The theory assumes that shear stresses in granular media is the same as in homogenous solid materials. Hence, the pile is in equilibrium if all parts in the pile, satisfying

$$\tau<\mu*\sigma, \qquad (1)$$

where τ is the shear stress, and, σ, the normal stress.

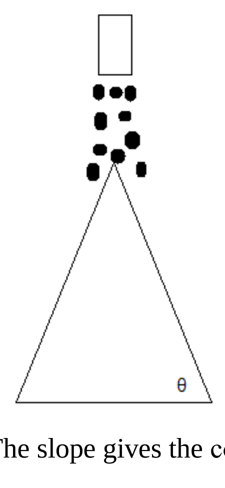


Fig.1 Cohesive-less heap at the base. The slope gives the coefficient of internal friction $\mu = \tan(\theta)$.

However, in paper [1], some granular materials show the profile of their piles is globally deviated from the ideal cone. Here we provide a model for the shape of a heap of cohesion-less powder with the varying density at different points in the heap, instead of the assumption of homogenous density in Coulomb material theory. It is obvious that assuming the changing density in the heap is closer to the nature of heaps. Let us focus on 2D semi-symmetry model.

**2D semi-symmetric model**

Let the height of a pile, h, be a function of x, a distance from the center of the pile (see Fig.2). Then, in the stability condition, the shear stresses and normal stresses in any inclined plane in the pile can be represented as functions in terms of the bulk density of the pile, $\rho_b$, the angle of the inclined plane,$\alpha$,ranging from 0 to arctan(-h'(x)), and the height of the pile, h(x). We assume that the porosity of the pile,$\Phi(z)$, is the function of depth through the Athy equation in reference [2]:

$$\Phi(z) = \Phi_0 e^{-kz}, \quad (2)$$

where $\Phi_0$ is the surface porosity, $k$ is the compaction coefficient and $z$ is the depth. The definition of porosity is given by

$$\Phi(z) = 1 - \frac{\rho_b}{\rho_p}, \quad (3)$$

where $\rho_p$ is the particle density of the pile and $\rho_b$ the bulk density of the pile. From Eq. (2) and Eq. (3), we have the bulk density relates to the particle density through the equation:

$$\rho_b = \left(1 - \Phi_0 e^{-kz}\right)\rho_p. \qquad (4)$$

Therefore, the weight of the powder on the inclined plane of the slope angle $\alpha$, (see Fig.2), is given by

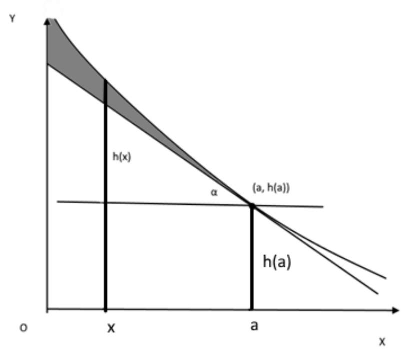


Fig.2 Sketch of the heap profile. P is the weight above the plane tilted by $\alpha$ from the horizontal and joining the free surface at the distance, a, from the center. The h(x) is the height of the heap at a position x from the center.

$$P = \int_{-\tan(\alpha)(x-a)+h(a)}^{h(x)} \int_0^a \left(1 - \Phi_0 e^{-k(h(x)-y)}\right)\rho_p g\,dx\,dy, \qquad (5)$$

where g is the gravity acceleration. The shear and normal stresses, $\tau$ and $\sigma$ in the stability condition (1) due to the weight on the plane are given:

$$\tau = \frac{P}{a}\sin(\alpha)\cos(\alpha) \quad and \quad \sigma = \frac{P}{a}cos^2(\alpha). \qquad (6)$$

Substituting the equation (5) into the equation (6) and simplifying, we obtain

$$\frac{\tau}{\rho_p\, g} = \left(\frac{D(a)}{2a} - \frac{\Phi_0}{2k} + \frac{\Phi_0}{2ka}F(a,\alpha)\right)\sin(2\alpha) - \frac{a}{4} + \frac{a}{4}\cos(2\alpha) \; (7)$$

$$\frac{\sigma}{\rho_p\, g} = \left(\frac{D(a)}{2a} - \frac{\Phi_0}{2k} + \frac{\Phi_0}{2ka}F(a,\alpha)\right)\cos(2\alpha) + \frac{D(a)}{2a} - \frac{a}{4}\sin(2\alpha), \quad (8)$$

where, $D(a) = \int_0^a \left(h(x) - h(a)\right)dx$, $F(a,\alpha) = \int_0^a e^{-k(h(x)+\tan(\alpha)(x-a)-h(a))}\,dx$. Eq.(7) and Eq.(8) form a parametric representation of the curve in the$\tau\sigma - plane$ with the parameters, a and $\alpha$. By rearranging, squaring, and adding (7) and (8), we obtain the equation,

$$\left(\frac{\tau}{\rho_p g}+\frac{a}{4}\right)^2+\left(\frac{\sigma}{\rho_p g}-\frac{D(a)}{2a}\right)^2=\left(\frac{D(a)}{2a}-\frac{\Phi_0}{2k}+\frac{\Phi_0}{2ka}F(a,\alpha)\right)^2+\left(\frac{a}{4}\right)^2. \quad (9)$$

When the weight of the powder P= 0, it led to the shear stress, $\tau = 0$, and the normal stress, $\sigma = 0$, the curve (9) pass the original point (0,0) under the condition

$$P=\int_{-\tan(\alpha)(x-a)+h(a)}^{h(x)}\int_0^a\left(1-\Phi_0 e^{-k(h(x)-y)}\right)\rho_p g\,dx\,dy=0 \quad (10)$$

Simplifying Equation (10), we obtain

$$h(a)=-\tan(\alpha)\,a+\left(\frac{1-\Phi_0}{k}\right)In(a). \quad (11)$$

The slope of the tangent line of the curve (9) at the original point, (0, 0), by using Eq. (6), is

$$\tau'_\sigma|_{(0,0)}=\frac{\tau'_\alpha}{\sigma'_\alpha}|_{(0,0)}=\tan(\alpha).$$

In fig, the tangent line at (0, 0) cuts $\tau\sigma-plane$ into two parts, with the property of one part, $\tau<\tan(\alpha)*\sigma$, under the tangent line, and other part, $\tau>\tan(\alpha)*\sigma$, up the tangent line. Therefore, if we assume that $\tan(\alpha)=\mu$, then $\tau<\mu*\sigma$, for all $\tau$ and $\sigma$, because the curve falls to the part below the tangent line. Substituting, $\tan(\alpha)$ with $\mu$, and a with x in Eq. (11), we obtain the mathematical expression for the shape of a non-cohesive granular heap as

$$h(x)=-\mu x+\left(\frac{1-\Phi_0}{k}\right)In(x),\quad x>0 \quad (12)$$

**Discussion**

The h(x) is approaching negative infinity as x is close to zero. In practice, we have to cut off certain distance due to the size of powder or the diameter of distributor. It is obvious to test that if the shape of heap is

$$h(x)=-\mu x+\left(\frac{1-\Phi_0}{k}\right)In(x),\quad x>\frac{1-\Phi_0}{k\mu}, \quad (13)$$

The heap is in equilibrium by using Coulomb's method of wedges. There are two terms in Eq. (12) where the first term represents a perfect conical shape and the second term refers to as the impact of various density in the heap.

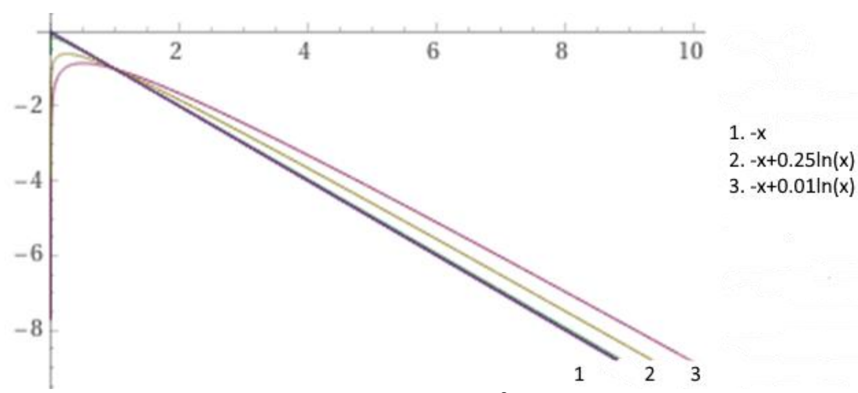


Fig.3 Examples of the heap profiles with $\theta=45^o$, $\varphi_o=0.5$, k = ∞, 2,10, respectively.

The impact of varying density causes a convex shapeinstead of a perfect conical shape. In addition, in Fig.3 we can see that the slope of h(x) increases to from zero to $\mu$as x increases from $\frac{1-\Phi_0}{k\mu}$ to infinity. As the compactions coefficient, k, increases to infinity, porosity reduces to zero, the bulk density is close to the density of particle, and the profile of h(x) is approaching to $h(x)=-\mu x$.

**Experimental observations**

This shape is exactly in agreement with the experiments in the paper [1, 5]. Michael Rackl, etc, were to investigate if the cone assumption is valid for three distinct bulk materials. They found out some shapes of the heaps show global deviation from the ideal cone shape. For example, the profile of the heap of corn grans has the character provided by our proposed model in Fig.4.

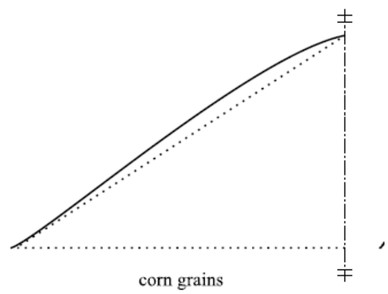


Fig.4 The heap profile of corn grains from reference [1].

**Concluding remarks**

The analytical expression is provided for the profile of a 2D-symmetric cohesion-less pile. With considering varying porosity at any point in the pile. The shape of the pile deviates an ideal cone shape. The results explain the factors to this deviation. This model can be applied to more accurately extract the quantitative information for the coefficient of internal friction, angle of repose, flowability of powder, porosity, and compaction coefficient then current models.

***Acknowledgement***

This study received funding by DOE (DE-NA0003867, 270136C).

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