On the Bayesian Estimation for two Component Mixture of Maxwell Distribution, Assuming Type I Censored Data

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Abstract

Mixture models constitute a finite and infinite number of components that explain different datasets. However there are many situations where mixture models comprise an interesting sketch of different aspects. In this study we explore the idea of mixture density under Type I censoring scheme. We model a heterogeneous population by means of two components mixture of the Maxwell distribution. The parameters of the Maxwell mixture are estimated and compared using the Bayes estimates under the square error loss function and precautionary loss function. A censored mixture data is simulated by probabilistic mixing for the computational purpose. Closed form expressions for the Bayes estimators and posterior risk are derived for the censored sample as well as for the complete sample. Some interesting comparison and properties of the estimates are observed and presented. A real life data application has also been discussed.

Keywords: Finite mixture of Maxwell distribution; Censored sampling; Fixed termination time; Limiting expression; Elicitation of Hyperparameters; Squared error loss function; Precautionary loss function.

1. Introduction

The Maxwell distribution is a probability distribution with application in physics and chemistry. The most frequent application is in the field of statistical mechanics. The temperature of any (massive) physical system is the result of the motions of the molecules and atoms which make up the system. These particles have a range of different velocities, and the velocity of any single particle constantly changes due to collisions with other particles. However, the fraction of a large number of particles within a particular velocity range is nearly constant. Then Maxwell distribution of velocities specifies this fraction, for any velocity range as a function of the temperature of the system. Tyagi and Bhattacharya (1989a, b) considered Maxwell distribution as a lifetime model for the first time. They obtained Bayes estimates and minimum variance unbiased estimators of the parameter and reliability function for the Maxwell distribution. Chaturvedi and Rani (1998) generalized Maxwell distribution and they obtained Classical and Bayesian estimators for generalized distribution. Bekker and Roux (2005) studied Empirical Bayes estimation for Maxwell distribution. These studies give mathematical handling to Maxwell distribution but ignore the application aspect of the Maxwell distribution.

The motivation of using mixture model is that in current scenario, analysts are able to describe estimates, predict and infer about the complex system of interest using more powerful and complex computational methods. But mixture model comprises an interesting sketch of all these aspects. Mixture models constitute a finite and infinite number of components that explain different datasets. The Bayesian approach to analyze mixture models has developed great interest between analysts. Posterior distribution is the workbench of Bayesian statisticians. It is obtained when prior information is combined with likelihood. Therefore prior information is necessary for Bayesian approach.

The prior information is purely subjective assessment of an expert before any data have been observed. Also Berger (1985) argue that when information is not in compact form the Bayesian analysis using non-informative priors or single most suitable consideration. *So in this study we consider two non-informative priors and two informative priors for comparison purpose*.

A finite mixture of some suitable probability distributions are recommended to study a population that is supposed to comprise a number of subpopulation mixed in an unknown proportion. A population of lifetimes of certain electrical elements may be divided into a number of subpopulations depending upon the possible case. Mixture model have been used in physical chemical, social science, biological and other fields. For example Sinha (1998) considered the Bayesian counterpart of the maximum likelihood estimates of the Mendenhall and Hader (1958) mixture. Saleem and Aslam (2008) use the Bayesian Analysis for the two components Mixture of the Rayleigh distribution assuming the uniform and Jeffery priors. A type–I mixture is stated as the mixture of probability density function from the same family, while a mixture of density functions from several families is called a type–II mixture. Now if we talk about the practical situations, a mixture population may have the known component densities and we need to infer only about the mixing weights. On the other hand, in many real life applications, there are known functional forms of component densities with unknown parameters but mixing weights are known and vice versa. *In this paper, type–I mixture models with unknown parameters of the known number of component densities belonging to the same parametric family and with unknown mixing weights are considered.*

Censoring is an unavoidable feature of the lifetime applications and is a form of missing data problem. An account of censoring can be seen in Leemis (1955), Deitz et al (1973), Klein (2009), Kalbfleisch et al (2002) and Smith (2002) that are valuable contribution to survival analysis technique for censored and truncated data. Jiang (1992) deals with maximum likelihood estimates using censored data for mixed Weibull distribution while Wang et al (1958) considers the estimators for survival function when censoring times are known. Censoring is divided into three types, i.e., left, right, and interval censoring is said to be employed if lifetime of an object is greater than an independent random number. In type–I (type–II) right censoring, the life-test termination time (the number of dead objects) is pre-specified. In ordinary type–I right censoring; the life-test termination time is the same for all the objects. The life-time of an object is called interval censored if it is known to fall in a known time-interval. *In our study, an ordinary type–I right censoring is considered with a fixed-test termination time.*

Now to answers the questions that why we use inverted gamma and inverted Chi square priors for mixture analysis and why we use censoring on data why not on parameter, As we know that the Maxwell model is skewed so we should have a prior which reflect expert knowledge in a better form so there should be a skewed prior for this model. Since Inverted Gamma is a natural conjugate prior for the Maxwell model therefore we use it. Then we chi square prior which is another form of the inverted gamma distribution in order to check that may be it perform better than inverted Gamma prior. If we talk about censoring then censoring is a data property and we cannot apply on parameters.

In this paper, random observations taken from this population are supposed to be characterized by one of the two distinct unknown members of a Maxwell distribution. So the two component mixture of the Maxwell distribution is recommended to model this population. Right censoring is considered and the observations greater than the fixed cut off censor value, T are taken as censored ones. The Inverse Transform method of simulation, and the computations involved are conducted using the packages Minitab, Mathematica, SAS and Excel. We may break this study in to following sections. The Maxwell mixture model is defined in Section 2 and its likelihood is developed in Section 3. Section 4,5 and 6, evaluate the Bayes estimators and their posterior risk under square error loss function and precautionary loss function using uniform, Jeffreys, Inverted Gamma and Inverted Chi-Squared prior. Elicitation of hyperparameters method is discussed in Section 7. Limiting expressions are derived in Section 8. In Section 9, the simulation study is performed. Section 10 presents the real life data which are used for the evaluation of Bayes estimates. Some concluding remarks are given in last Section 11.

2. The Mixture Model

A finite mixture density function with the two component densities of specified parametric form with unknown mixing weights $(p, 1-p)$ is defined as follows

$$g(x) = pf_1(x) + (1-p)f_2(x), 0 < p < 1 \quad (1)$$

The following Maxwell distribution is assumed for both components of the mixture.

$$\mathrm{f}_i\left(x\right)=\frac{4}{\sqrt{\pi}}\frac{1}{\theta_i^{\frac{3}{2}}}x^2e^{-\frac{x^2}{\theta_i}}\ ,i=1,2\ ,\theta_i>0\ ,\ 0\le x\le\infty$$

so the mixture model takes the following form

$$g\left(x\right)=p\left(\frac{4}{\sqrt{\pi}}\frac{1}{\theta_1^{\frac{3}{2}}}x^2e^{-\frac{x^2}{\theta_1}}\right)+q\left(\frac{4}{\sqrt{\pi}}\frac{1}{\theta_2^{\frac{3}{2}}}x^2e^{-\frac{x^2}{\theta_2}}\right)\ ;q=1-p,0<p<1$$

Krishna and Malik (2009) use the following form of distribution function:

$$G(x)=\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{x^2}{\theta},\frac{3}{2}\right)$$

where $\Gamma\left(x,a\right)=\int_0^x e^{-u}u^{a-1}du$, is the incomplete gamma function.

So the corresponding distribution function is given by

$$F(x)=pG_1(x)+(1-p)G_2(x)$$

$$F(x)=p\left\{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{x^2}{\theta_1},\frac{3}{2}\right)\right\}+(1-p)\left\{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{x^2}{\theta_2},\frac{3}{2}\right)\right\}$$

3. Likelihood Function

Suppose n units from the above mixture model are used to life testing experiment. Let the test be conducted and it is observe that out of n test is terminated as soon as the rth failure occurs and the remaining $n-r$ units are still working. As in Mendenhall and Hader (1958) enlighten that in many real life situations only the failed objects can easily be identified as member of either subpopulation 1 or subpopulation 2 . So, depending upon the cause of failure it may be observed that r_1 and r_2 are identified as members of the first and second subpopulation respectively. It is apparent that $r=r_1+r_2$ and remaining $n-r$ objects provide no information about the subpopulation to which they belong. We define, x_{ij} as the failure time of the jth subpopulation, where $j=1,2,3,.......;\ i=1,2;\ 0<x_{1j},x_{2j}\le\infty$. So the likelihood function for the given condition is:

$$L(\theta_1,\theta_2,p\,|\,x)\propto\left\{\prod_{j=1}^{r_1}pf_1\left(x_{1j}\right)\right\}\left\{\prod_{j=1}^{r_2}qf_2\left(x_{2j}\right)\right\}\left\{\left(1-F\left(t\right)\right)^{n-r}\right\} \qquad (2)$$

where $\mathbf{x}=\left(x_{11},x_{12}......,x_{1r_1},x_{21},x_{22},.....,x_{2r_2}\right)$ is the observed failure times for the non-censored observations

$$L(\theta_1,\theta_2,p\,|\,\mathbf{x})\propto\left\{\prod_{j=1}^{r_1}p\left(\frac{4}{\sqrt{\pi}}\frac{1}{\theta_1^{\frac{3}{2}}}x^2e^{-\frac{x^2}{\theta_1}}\right)\right\}\left\{\prod_{j=1}^{r_2}q\left(\frac{4}{\sqrt{\pi}}\frac{1}{\theta_2^{\frac{3}{2}}}x^2e^{-\frac{x^2}{\theta_2}}\right)\right\}\left\{\left(1-F\left(t\right)\right)^{n-r}\right\},$$

$$1-F(t)=1-\frac{p}{0.5\sqrt{\pi}}\Gamma\left(\frac{t^2}{\theta_1},\frac{3}{2}\right)+\frac{q}{0.5\sqrt{\pi}}\Gamma\left(\frac{t^2}{\theta_2},\frac{3}{2}\right),\text{ where } t_1=t_2=t$$

$$L(\theta_1,\theta_2,p\,|\,\mathbf{x})\propto\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}p^{r_1+k-m}q^{r_2+m}\left(\frac{4}{\sqrt{\pi}}\right)^{r_1}\left(\frac{4}{\sqrt{\pi}}\right)^{r_2}\prod_{j=1}^{r_1}x_{1j}{}^2\prod_{j=1}^{r_2}x_{2j}{}^2\times$$
$$\frac{1}{\theta_1^{C_1}}\exp\left(-\frac{A_1}{\theta_1}\right)\frac{1}{\theta_2^{C_2}}\exp\left(-\frac{A_2}{\theta_2}\right)D_1^{k-m}D_2{}^m, \qquad (3)$$

where $D_i=\frac{1}{0.5\sqrt{\pi}}\Gamma\left(\frac{t^2}{\theta_i},\frac{3}{2}\right),A_i=\sum_{j=1}^{r_i}x_{ij}{}^2,i=1,2$ and $C_i=\frac{3r_i}{2},i=1,2$

4. Bayesian Estimation using Uninformative Priors

Since Bayesian estimation can be applied even when no prior information is available, so we can say that uninformative prior is a prior which contain no information about parameter θ. Among the techniques that have been proposed for determining uninformative priors, Jeffreys (1961) suggests the most widely use method. Box and Taio (1973), define an uninformative prior as prior which provides little information relative to the experiment. Bernardo and Smith (1994) use a similar definition; they say that uninformative priors have minimal effect relative to the data, on the final inference. They regard the uninformative prior as a mathematical tool; it is no a uniquely uninformative prior. Bernardo (1979b) argue that an uninformative prior should be regarded as reference prior, i.e. a prior which is convenient to use as a standard in analyzing statistical data. Geisser (1984) also proposed some techniques for uninformative priors. The most common examples of uninformative priors are Uniform and Jeffreys. Both priors are used only when no formal prior information is available.

4.1. The Uniform (Uninformative) Prior

Bayes (1763), Laplace (1812) and Geisser (1984) suggest that one may take uniform distribution for the unknown parameter θ in the absence of sufficient reason for assigning unequal probabilities to the values in the parameter space had created a lot of discomforts for the users of Bayes theorem for inferential purposes. Uniform priors are particularly easy to specify in the case of a parameter with bounded support. The simplest situation to consider is when θ is finite.

Let $\theta_1 \sim Uniform\ \forall\ \theta_1 \in (0,\infty)$, $\theta_2 \sim Uniform\ \forall\ \theta_2 \in (0,\infty)$ and $p \sim U(0,1)$ .Assuming independence , we have improper joint prior that is proportion to a constant which is incorporated with the above likelihood (4) and we have joint posterior and marginal distributions. The joint posterior distribution using uniform prior as follows

$$p(\theta_1,\theta_2,p\,|\,\mathbf{x}) \propto \sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}p^{r_1+k-m}q^{r_2+m}\left(\frac{4}{\sqrt{\pi}}\right)^{r_1}\left(\frac{4}{\sqrt{\pi}}\right)^{r_2}\prod_{j=1}^{r_1}x_{1j}^{\ 2}\prod_{j=1}^{r_2}x_{2j}^{\ 2}\times$$
$$\frac{1}{\theta_1^{C_1}}\exp\left(-\frac{A_1}{\theta_1}\right)\frac{1}{\theta_2^{C_2}}\exp\left(-\frac{A_2}{\theta_2}\right)D_1^{\,k-m}D_2^{\,m},\ 0<\theta_1<\infty, 0<\theta_2<\infty, 0<p<1 \qquad (4)$$

4.2 Bayes Estimators using Uniform Prior

Bayes estimator is an estimator or decision rule that maximizes the posterior expected value of utility function or minimizes the posterior expected value of the loss function. The loss function is the real valued function that clearly provides a loss for decision a given parameter θ. The square error loss function (SELF) $L\left(\theta,a^*\right)=\left(\theta-a^*\right)^2$ was proposed by Legendre (1805) and Gauss (1810) to develop least square theory. Later it was used in estimation problem when unbiased estimators of θ were evaluated in terms of the risk function $R(\theta,g)$ which become nothing but the variance of the estimators. Norstrom (1996) introduced an alternative asymmetric precautionary loss function (PLF), and also presented a general class of precautionary loss functions as a special case which is defined as $L_4 = L(\lambda,d)=\frac{(\lambda-d)^2}{d}$, Bayes estimator using this loss function is $d^* = \sqrt{E(\lambda^2\,|\,\mathbf{x})}$ and $E_{\lambda|\mathbf{x}}L(\lambda,d)=2\left(\sqrt{E(\lambda^2\,|\,\mathbf{x})}-E(\lambda\,|\,\mathbf{x})\right)$ is posterior risk. The respective marginal distribution yield the following Bayes estimators of θ_1,θ_2 and of p under the squared error loss function.

$$E(\theta_1\,|\,\mathbf{x})=\frac{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1-2\right)}{A_1^{c_1-2}}\frac{\Gamma\left(c_2-1\right)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty\theta_1 D_1^{\,k-m}D_2^{\,m}d\theta_1 d\theta_2}{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1-1\right)}{A_1^{c_1-1}}\frac{\Gamma\left(c_2-1\right)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty D_1^{\,k-m}D_2^{\,m}d\theta_1 d\theta_2} \qquad (5)$$

$$E(\theta_2\,|\,\mathbf{x})=\frac{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1-1\right)}{A_1^{c_1-1}}\frac{\Gamma\left(c_2-2\right)}{A_2^{c_2-2}}\int_0^\infty\int_0^\infty\theta_2 D_1^{\,k-m}D_2^{\,m}d\theta_1 d\theta_2}{\displaystyle\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1-1\right)}{A_1^{c_1-1}}\frac{\Gamma\left(c_2-1\right)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty D_1^{\,k-m}D_2^{\,m}d\theta_1 d\theta_2} \qquad (6)$$

$$E(p|\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B(\alpha_3,\alpha_2)\frac{\Gamma(c_1-1)}{A_1^{c_1-1}}\frac{\Gamma(c_2-1)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B(\alpha_1,\alpha_2)\frac{\Gamma(c_1-1)}{A_1^{c_1-1}}\frac{\Gamma(c_2-1)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2} \quad (7)$$

where $i=1,2$. $\alpha_1=r_1+k-m+1$, $\alpha_2=r_2+m+1$, $\alpha_3=r_1+k-m+2$ and $\alpha_4=r_1+k-m+3$. Similarly Bayes estimators using precautionary loss function can also be derived accordingly.

4.3. Posterior Risks assuming Uniform Prior

The posterior risks of θ_1,θ_2 and p using the uniform prior are given as

$$\rho(\theta_1|\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B(\alpha_1,\alpha_2)\frac{\Gamma(c_1-3)}{A_1^{c_1-3}}\frac{\Gamma(c_2-1)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty \theta_1^2 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B(\alpha_1,\alpha_2)\frac{\Gamma(c_1-1)}{A_1^{c_1-1}}\frac{\Gamma(c_2-1)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E(\theta_1|\mathbf{x})\right\}^2 \quad (8)$$

$$\rho(\theta_2|\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B(\alpha_1,\alpha_2)\frac{\Gamma(c_1-1)}{A_1^{c_1-1}}\frac{\Gamma(c_2-3)}{A_2^{c_2-3}}\int_0^\infty\int_0^\infty \theta_2^2 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B(\alpha_1,\alpha_2)\frac{\Gamma(c_1-1)}{A_1^{c_1-1}}\frac{\Gamma(c_2-1)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E(\theta_2|\mathbf{x})\right\}^2 \quad (9)$$

$$\rho(p|\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B(\alpha_4,\alpha_2)\frac{\Gamma(c_1-1)}{A_1^{c_1-1}}\frac{\Gamma(c_2-1)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B(\alpha_1,\alpha_2)\frac{\Gamma(c_1-1)}{A_1^{c_1-1}}\frac{\Gamma(c_2-1)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E(p|\mathbf{x})\right\}^2 \quad (10)$$

where $\left\{E(\theta_1|\mathbf{x})\right\}$, $\left\{E(\theta_2|\mathbf{x})\right\}$ and $\left\{E(p|\mathbf{x})\right\}$ are given is equations (5),(6) and (7) respectively.

Note that here all integrals are evaluated numerically using Mathematica 6.0. The posterior risks under precautionary loss function can also be derived in similar manner.

5. The Jeffreys (Uninformative) Prior

Jeffreys prior is another form of uninformative prior which is also called reference prior, it is based on Fisher information matrix habitually lead to a family of improper priors. Under some regularity conditions specially in case of one parameter it does not reveal lack of knowledge, Jeffreys' prior illustrate the sort of prior knowledge which would make the data as posterior dominant as possible. The posterior distribution based on Jeffreys' prior may then be used as a benchmark or a reference for the class of posterior distribution which may be obtained from other priors. For the Maxwell model, given in Section 2, let the Jeffrey priors $g(\theta_i)\propto\sqrt{|I(\theta_i)|}$, $I(\theta_i)=-E\left[\frac{\partial^2 f_i(x|\theta_i)}{\partial\theta_i^2}\right]$ where $i=1,2$ are $g(\theta_1)\propto\frac{1}{\theta_1}0<\theta_1<\infty$ $g(\theta_2)\propto\frac{1}{\theta_2}0<\theta_2<\infty$ and $g_3(p)=1,\ 0<p<1$. By assuming independence we obtain a joint prior $g(\theta_1,\theta_2,p)\propto\frac{1}{\theta_1\theta_2}$ which is incorporated with the likelihood (4) to yield the joint posterior and marginal distributions. So the Joint posterior distribution is given as

$$P(\theta_1,\theta_2,p|\mathbf{x})\propto\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}p^{r_1+k-m}q^{r_2+m}\left(\frac{4}{\sqrt{\pi}}\right)^{r_1}\left(\frac{4}{\sqrt{\pi}}\right)^{r_2}\prod_{j=1}^{r_1}x_{1j}^{\ 2}\prod_{j=1}^{r_2}x_{2j}^{\ 2}\times$$
$$\frac{1}{\theta_1^{C_1+1}}\exp\left(-\frac{A_1}{\theta_1}\right)\frac{1}{\theta_2^{C_2+1}}\exp\left(-\frac{A_2}{\theta_2}\right)D_1^{k-m}D_2^{m},\ 0<\theta_1<\infty,0<\theta_2<\infty,0<p<1 \quad (11)$$

5.1 Bayes Estimators using the Jeffrey Prior

The respective marginal distribution yield the following Bayes estimators of θ_1,θ_2 and of p under the squared error loss function.

$$E(\theta_1\,|\,\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1-1\right)}{A_1^{c_1-1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty \theta_1 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2} \quad (12)$$

$$E(\theta_2\,|\,\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2-1\right)}{A_2^{c_2-1}}\int_0^\infty\int_0^\infty \theta_2 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2} \quad (13)$$

$$E(p\,|\,\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_3,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2} \quad (14)$$

5.2. Posterior Risk assuming Jeffrey Prior

Under SELF the posterior risks of θ_1,θ_2 and p using Jeffreys' prior are provided as

$$\rho(\theta_1\,|\,\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1-2\right)}{A_1^{c_1-2}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty \theta_1 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E\left(\theta_1\,|\,\mathbf{x}\right)\right\}^2 \quad (15)$$

$$\rho(\theta_2\,|\,\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2-2\right)}{A_2^{c_2-2}}\int_0^\infty\int_0^\infty \theta_2 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E\left(\theta_2\,|\,\mathbf{x}\right)\right\}^2 \quad (16)$$

$$\rho(p\,|\,\mathbf{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_3,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}B\left(\alpha_1,\alpha_2\right)\frac{\Gamma\left(c_1\right)}{A_1^{c_1}}\frac{\Gamma\left(c_2\right)}{A_2^{c_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E\left(p\,|\,\mathbf{x}\right)\right\}^2 \quad (17)$$

where $\left\{E\left(\theta_1\,|\,\mathbf{x}\right)\right\},\left\{E\left(\theta_2\,|\,\mathbf{x}\right)\right\}$ and $\left\{E\left(p\,|\,\mathbf{x}\right)\right\}$ are given is equations (12),(13) and (14) respectively. The posterior risk under precautionary loss function using Jeffrey prior can also be derived as SELF.

6. Informative Priors

An informative prior expresses specific, definite information about a variable. The terms "prior" and "posterior" are generally relative to a specific datum or observation. In case of an informative prior, the use of prior information is equivalent to adding a number of observations to a given sample size, and therefore leads to a reduction of the variance or posterior risk of the Bayes estimates.

6.1. The Conjugate Prior

In probability theory and statistics, the Conjugate prior (Inverted Gamma distribution) is a two-parameter family of continuous probability distribution on the positive real line, which is the distribution of the reciprocal of a variable distributed according to the gamma distribution.

Let $\theta_1 \sim \textit{Inverted Gamma}(a_1,b_1)$, $\theta_2 \sim \textit{Inverted Gamma}(a_2,b_2)$ and $P \sim U(0,1)$,assuming independence, we have a joint prior $g(\theta_1,\theta_2,p)\propto \frac{b_1^{a_1}}{\Gamma(a_1)}\frac{1}{\theta_1^{a_1+1}}e^{-\frac{b_1}{\theta_1}}.\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{1}{\theta_2^{a_2+1}}e^{-\frac{b_2}{\theta_2}}$ which is incorporated with the likelihood given in (4).

$$p(\theta_1,\theta_2,p\,|\,\mathrm{x}) \propto \sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m} p^{r_1+k-m} q^{r_2+m} \frac{1}{\theta_1^{\frac{3r_1}{2}+a_1+1}} \exp\left(-\left(\sum_{j=1}^{r_1} x_{1j}^{\,2}+b_1\right)/\theta_1\right)\times$$

$$\frac{1}{\theta_2^{\frac{3r_2}{2}+a_2+1}} \exp\left(-\left(\sum_{j=1}^{r_2} x_{2j}^{\,2}+b_2\right)/\theta_2\right)\left(D_1\right)^{k-m}\left(D_2\right)^{m},\quad \theta_1,\theta_2>0, 0<p<1 \qquad (18)$$

The marginal distributions with respect to parameters q_1, q_2 and p are elaborated as

$$p(\theta_1\,|\,\mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{1}{\theta_1^{c_1+a_1+1}} e^{-\left(\frac{A_1+b_1}{\theta_1}\right)} \frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}, \qquad (19)$$

$$p(\theta_2\,|\,\mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{1}{\theta_1^{c_2+a_2+1}} e^{-\left(\frac{A_2+b_2}{\theta_2}\right)} \frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}, \qquad (20)$$

$$p(p\,|\,\mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m} p^{r_1+k-m}q^{r_2+m}\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}. \qquad (21)$$

6.1.1. Bayes Estimators using the Conjugate Prior

Bayes estimator is an estimator or decision rule that maximizes the posterior expected value of utility function or minimizes the posterior expected value of the loss function. The respective marginal distribution yield the following Bayes estimators of q_1, q_2 *and* p under the squared error loss function.

$$E(\theta_1\,|\,\mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1-1)}{(A_1+b_1)^{c_1+a_1-1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty \theta_1 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}, \qquad (22)$$

$$E(\theta_2\,|\,\mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2-1)}{(A_2+b_2)^{c_2+a_2-1}}\int_0^\infty\int_0^\infty \theta_2 D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}, \qquad (23)$$

$$E(p\,|\,\mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}. \qquad (24)$$

Where $D_i, A_i,$ and defined above, while a_i, $i = 1,..,4$ are defined as

$\alpha_1 = r_1+k-m+1, \alpha_2 = r_2+m+1, \alpha_3 = r_1+k-m+2$ and $\alpha_4 = r_1+k-m+3$.

6.1.2. Posterior Risk using the Conjugate Prior

When presenting a Statistical estimate, it is necessary to indicate the accuracy of the estimates. The Bayesian measure of the accuracy of an estimate is the posterior risk of the estimate. The expressions of the Bayes posterior risks under square error loss function are

$$\rho(\theta_1 | \mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1-2)}{(A_1+b_1)^{c_1+a_1-2}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+a_2}}\int_0^\infty\int_0^\infty \theta_1^2 D_1^{k-m}D_2^m d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_1)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^m d\theta_1 d\theta_2} - \left\{E(\theta_1 | \mathrm{x})\right\}^2 \quad (25)$$

$$\rho(\theta_2 | \mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2-2)}{(A_2+b_2)^{c_2+a_2-2}}\int_0^\infty\int_0^\infty \theta_2^2 D_1 D_2 d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_1)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1 D_2 d\theta_1 d\theta_2} - \left\{E(\theta_2 | \mathrm{x})\right\}^2 \quad (26)$$

$$\rho(p | \mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_3,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_2)^{c_2+b_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^m d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+a_1)}{(A_1+b_1)^{c_1+a_1}}\frac{\Gamma(c_2+a_2)}{(A_2+b_1)^{c_2+a_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^m d\theta_1 d\theta_2} - \left\{E(p | \mathrm{x})\right\}^2 \quad (27)$$

where $E(q_1 | \mathrm{x})$, $E(\theta_2 | \mathrm{x})$ and $E(p | \mathrm{x})$ are specified in equations (22), (23) and (24) respectively. Here we use numerical integration in order to evaluate integrals given in variances equations. Similarly we can estimate the Bayes estimators and Bayes posterior risks under precautionary loss function using Conjugate prior.

6.2. The Inverted Chi-square Prior

The inverted chi-square distribution is the distribution of a random variable whose multiplicative (reciprocal) has a chi-square distribution. It is also often defined as the distribution of a random variable whose reciprocal divided by its degrees of freedom is a chi-square distribution. Let $\theta_1 \sim \textit{Inverted Chi-squre}(a_1, b_1)$, $q_2 : \textit{Inverted Chi- squre}(a_2, b_2)$ and $P \sim U(0,1)$ assuming independence, we have a joint priors,

$$g(\theta_1,\theta_2,p) = \frac{b_1^{\frac{a_1}{2}}}{2^{\frac{a_1}{2}}\Gamma\left(\frac{a_1}{2}\right)\theta_1^{\frac{a_1}{2}+1}} e^{-\frac{b_1}{2\theta_1}} \cdot \frac{b_2^{\frac{a_2}{2}}}{2^{\frac{a_2}{2}}\Gamma\left(\frac{a_2}{2}\right)\theta_2^{\frac{a_2}{2}+1}} e^{-\frac{b_2}{2\theta_2}}$$,which is incorporated with the likelihood given in (4).

So the joint posterior distribution gets the following form

$$p(\theta_1,\theta_2,p | \mathrm{x}) \propto \sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m} p^{r_1+k-m} q^{r_2+m} \frac{1}{\theta_1^{c_1+\xi_1+1}} e^{-\left(\frac{A_1+\psi_1}{\theta_1}\right)} \frac{1}{\theta_2^{c_2+\xi_2+1}} e^{-\left(\frac{A_2+\psi_2}{\theta_2}\right)} D^{k-m} D^m . \quad (28)$$

where $\xi_i = \frac{a_i}{2}, \psi_i = \frac{b_i}{2},\ i = 1,2$.

Now the marginal distributions of the corresponding parameters q_1, q_2 and p are mentioned as

$$p(\theta_1 | \mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{1}{\theta_1^{c_1+\xi_1+1}} e^{-\left(\frac{A_1+\psi_1}{\theta_1}\right)}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty D_1^{k-m}D_2^m d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^m d\theta_2 d\theta_1}, \quad (29)$$

$$p(\theta_2 | \mathrm{x}) = \frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{1}{\theta_1^{c_2+\xi_2+1}} e^{-\left(\frac{A_2+\psi_2}{\theta_2}\right)}\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\int_0^\infty D_1^{k-m}D_2^m d\theta_1}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^m d\theta_2 d\theta_1}, \quad (30)$$

$$p(p|\mathrm{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}p^{r_1+k-m}q^{r_2+m}\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_2 d\theta_1}. \quad (31)$$

6.2.1. Bayes Estimators using the Inverted Chi-square Prior

The under the squared error loss function, respective marginal distribution yield the following Bayes estimators of θ_1,θ_2 and of p

$$E(\theta_1|\mathrm{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1-1)}{(A_1+\psi_1)^{c_1+\xi_1-1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty \theta_1 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}, \quad (32)$$

$$E(\theta_2|\mathrm{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2-1)}{(A_2+\psi_2)^{c_2+\xi_2-1}}\int_0^\infty\int_0^\infty \theta_2 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}, \quad (33)$$

$$E(p|x)=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_3,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}. \quad (34)$$

Where D_i, A_i, and a_i, $i=1,..,4$ are defined above.

6.2.2. Posterior Risk using the Inverted Chi-square Prior

The expressions for the variances of the Bayes estimators under square error loss function are

$$\rho(\theta_1|\mathrm{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1-2)}{(A_1+\psi_1)^{c_1+\xi_1-2}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty \theta_1^2 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E(\theta|\mathrm{x})\right\}^2$$

(35)

$$\rho(\theta_2|\mathrm{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2-2)}{(A_2+\psi_2)^{c_2+\xi_2-2}}\int_0^\infty\int_0^\infty \theta_2^2 D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E(\theta_2|\mathrm{x})\right\}^2$$

(36)

$$\rho(p|\mathrm{x})=\frac{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_4,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}{\sum_{k=0}^{n-r}\sum_{m=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{m}\beta(\alpha_1,\alpha_2)\frac{\Gamma(c_1+\xi_1)}{(A_1+\psi_1)^{c_1+\xi_1}}\frac{\Gamma(c_2+\xi_2)}{(A_2+\psi_2)^{c_2+\xi_2}}\int_0^\infty\int_0^\infty D_1^{k-m}D_2^{m}d\theta_1 d\theta_2}-\left\{E(p|\mathrm{x})\right\}^2$$

(37)

where $E(\theta_1|\mathrm{x})$, $E(\theta_2|\mathrm{x})$ and $E(p|\mathrm{x})$ are mentioned in equations (32), (33) and (34) respectively. Here we use numerical integration in order to evaluate integrals given in variance expressions. The Bayes estimator and posterior risk can also be derived under precautionary loss function using Chi-Squared priors.

7. Elicitation of Hyperparameters

According to Garthwaite et al. (2004) elicitation is the process of formulating a person's knowledge and beliefs about one or more uncertain quantities into a (joint) probability distribution for those quantities. In the context of Bayesian statistical analysis, it arises most usually as a method for specifying the prior distribution for one or more unknown parameters of a statistical model. Aslam (2003) purposed four methods for elicitation, three methods for two treatments and one method for general treatments. Parameters involve in the Bayes estimates and variances by using both inverted gamma prior and inverted chi-square prior are elicited according to method of elicitation via prior predictive approach which is also one of them, where prior predictive distributions using inverted gamma prior and inverted chi-square prior are derived by using following formula

$$p(y)=\int_0^\infty\int_0^\infty\int_0^1 p(\theta_1,\theta_2,p)p(y\mid\theta_1,\theta_2,p)\,d\theta_1 d\theta_2 dp\,.$$

According to the expert probabilities we consider four intervals for the elicitation, and the set of hyperparameters with minimum values are chosen to be the elicited values of the hyperparameters. The resultant prior predictive distributions for the mixture of Maxwell model are as follows

7.1 Elicitation using Conjugate prior

The prior predictive distribution equation using Conjugate prior is

$$p(y)=\frac{b_1^{a_1}}{\Gamma a_1}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(2,1)\frac{\Gamma\left(a_1+\frac{3}{2}\right)}{(b_1+y^2)^{a_1+\frac{3}{2}}}+\frac{b_2^{a_2}}{\Gamma a_2}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(1,2)\frac{\Gamma\left(a_2+\frac{3}{2}\right)}{(b_2+y^2)^{a_2+\frac{3}{2}}},\quad y>0 \qquad (38)$$

Since we have to elicit four parameters therefore we considered four intervals. The set of hyperparameters with minimum values are considered to be the elicited values of the hyperparameters. Using the prior predictive distribution given in (38), the experts' probabilities are assumed to be 0.12, 0.12, 0.12 and 0.12 which are associated with the intervals $0\le y\le 10$ and $10.1\le y\le 20$ $20.1\le y\le 30$ and $30.1\le y\le 40$ respectively as

$$\int_{0.1}^{10}\frac{b_1^{a_1}}{\Gamma a_1}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(2,1)\frac{\Gamma\left(a_1+\frac{3}{2}\right)}{(b_1+y^2)^{a_1+\frac{3}{2}}}+\frac{b_2^{a_2}}{\Gamma a_2}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(1,2)\frac{\Gamma\left(a_2+\frac{3}{2}\right)}{(b_2+y^2)^{a_2+\frac{3}{2}}}dy=0.12, \qquad (39)$$

$$\int_{10.1}^{20}\frac{b_1^{a_1}}{\Gamma a_1}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(2,1)\frac{\Gamma\left(a_1+\frac{3}{2}\right)}{(b_1+y^2)^{a_1+\frac{3}{2}}}+\frac{b_2^{a_2}}{\Gamma a_2}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(1,2)\frac{\Gamma\left(a_2+\frac{3}{2}\right)}{(b_2+y^2)^{a_2+\frac{3}{2}}}dy=0.12. \qquad (40)$$

$$\int_{20.1}^{30}\frac{b_1^{a_1}}{\Gamma a_1}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(2,1)\frac{\Gamma\left(a_1+\frac{3}{2}\right)}{(b_1+y^2)^{a_1+\frac{3}{2}}}+\frac{b_2^{a_2}}{\Gamma a_2}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(1,2)\frac{\Gamma\left(a_2+\frac{3}{2}\right)}{(b_2+y^2)^{a_2+\frac{3}{2}}}dy=0.12 \qquad (41)$$

$$\int_{30.1}^{40}\frac{b_1^{a_1}}{\Gamma a_1}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(2,1)\frac{\Gamma\left(a_1+\frac{3}{2}\right)}{(b_1+y^2)^{a_1+\frac{3}{2}}}+\frac{b_2^{a_2}}{\Gamma a_2}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(1,2)\frac{\Gamma\left(a_2+\frac{3}{2}\right)}{(b_2+y^2)^{a_2+\frac{3}{2}}}dy=0.12 \qquad (42)$$

For eliciting the hyperparameters a_1, a_2, b_1 and b_2, the equations (39) to (42) are simultaneously solved through the computer program developed in SAS package using the 'PROC SYSLIN' command and the values of the hyperparameters a_1, a_2, b_1 and b_2 are found to be 0.234861, 0.816093, 0.000059701 and 10.999744 respectively.

7.2 Elicitation using Inverted Chi-Square prior

The prior predictive distribution equation using inverted chi-squared prior is

$$p(y)=\frac{b_1^{\frac{a_1}{2}}}{2^{\frac{a_1}{2}}\Gamma\frac{a_1}{2}}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(2,1)\frac{\Gamma\left(\frac{a_1}{2}+\frac{3}{2}\right)}{(\frac{b_1}{2}+y^2)^{\frac{a_1}{2}+\frac{3}{2}}}+\frac{b_2^{\frac{a_2}{2}}}{2^{\frac{a_2}{2}}\Gamma\frac{a_2}{2}}\left(\frac{4}{\sqrt{\pi}}\right)y^2\beta(1,2)\frac{\Gamma\left(\frac{a_2}{2}+\frac{3}{2}\right)}{(\frac{b_2}{2}+y^2)^{\frac{a_2}{2}+\frac{3}{2}}},\quad y>0 \tag{43}$$

Using similar criteria as defined for inverted gamma prior, the values of the hyperparameters a_1, a_2, b_1 and b_2 are 1.217054, 0.332481, 0.00006621 and 14.814588 respectively.

8. Limiting expressions

Suppose $T\to\infty$, all observations which are slot in our analysis become uncensored, and consequently r tends to n, r_1 tends to the unknown n_1 and r_2 to the unknown n_2. Accordingly, the sum of information enclosed in the sample become increasing, as a result variances of the estimates become diminish. The expressions for the complete sample Bayes estimates and their variances are simplified as

Table 1: The Limiting Expression for the Bayes Estimators using Uniform and Jeffreys Priors

Parameters	*Bayes estimates (Uniform)*	*Bayes estimates(Jeffreys)*
θ_1	$\lim_{T\to\infty}\theta_1\mid\mathbf{x}=\frac{2(\sum_{j=1}^{n}x_{1j}^2)}{3n_1-4}$	$\lim_{T\to\infty}\theta_1\mid\mathbf{x}=\frac{2(\sum_{j=1}^{n}x_{1j}^2)}{3n_1-2}$
θ_2	$\lim_{T\to\infty}\theta_2\mid\mathbf{x}=\frac{2(\sum_{j=1}^{n}x_{2j}^2)}{3n_2-4}$	$\lim_{T\to\infty}\theta_2\mid\mathbf{x}=\frac{2(\sum_{j=1}^{n}x_{2j}^2)}{3n_2-2}$
p	$\lim_{T\to\infty}p\mid\mathbf{x}=\frac{n_1+1}{n+2}$	$\lim_{T\to\infty}p\mid\mathbf{x}=\frac{n_1+1}{n+2}$

Table 2: The Limiting Expression for the Bayes Posterior Risks using Uniform and Jeffreys Priors

Parameters	***Uniform Prior***	***Jeffreys priors***
θ_1	$\lim_{T\to\infty}\rho(\theta_1\mid\mathbf{x})=\frac{8(\sum_{j=1}^{n}x_{1j}^2)^2}{(3n_1-4)^2(3n_1-6)}$	$\lim_{T\to\infty}\rho(\theta_1\mid\mathbf{x})=\frac{8(\sum_{j=1}^{n}x_{1j}^2)^2}{(3n_1-2)^2(3n_1-4)}$
θ_2	$\lim_{T\to\infty}\varepsilon\rho(\theta_2\mid\mathbf{x})=\frac{8(\sum_{j=1}^{n}x_{2j}^2)^2}{(3n_2-4)^2(3n_2-6)}$	$\lim_{T\to\infty}\rho(\theta_1\mid\mathbf{x})=\frac{8(\sum_{j=1}^{n}x_{1j}^2)^2}{(3n_1-2)^2(3n_1-4)}$
p	$\lim_{T\to\infty}\rho(p\mid\mathbf{x})=\frac{(n_1+1)(n_2+1)}{(n+2)^2(n+3)}$	$\lim_{T\to\infty}\rho(p\mid\mathbf{x})=\frac{(n_1+1)(n_2+1)}{(n+2)^2(n+3)}$

Table 3: Bayes Estimators using informative priors and ML estimator as T ® ¥

Parameters	Bayes estimates (IG)*	Bayes estimates(IC)
θ_1	$E(\theta_1 \mid \mathrm{x}) = \frac{2(\sum_{j=1}^{n} x_{1j}^2 + b_1)}{(3n_1 + 2a_1 - 2)}$	$E(\theta_1 \mid \mathrm{x}) = \frac{(2\sum_{j=1}^{n} x_{1j}^2 + b_1)}{(3n_1 + a_1 - 2)}$
θ_2	$E(\theta_2 \mid \mathrm{x}) = \frac{2(\sum_{j=1}^{n} x_{2j}^2 + b_2)}{(3n_2 + 2a_2 - 2)}$	$E(\theta_2 \mid \mathrm{x}) = \frac{(2\sum_{j=1}^{n} x_{2j}^2 + b_2)}{(3n_2 + a_2 - 2)}$
p	$E(p \mid \mathbf{x}) = \frac{n_1 + 1}{n + 2}$	$E(p \mid \mathbf{x}) = \frac{n_1 + 1}{n + 2}$

f we use these estimators, our posterior risk will be small because we are using complete information of data as compared to censored one.

Table 4: Limiting Expression for the Posterior risks of informative priors as T ® ¥

Parameters	***Conjugate Prior***	***Chi Square prior***
θ_1	$\rho(\theta_1 \mid \mathbf{x}) = \frac{8(\sum_{j=1}^{n} x_{1j}^2 + b_1)^2}{(3n_1 + 2a_1 - 2)^2(3n_1 + 2a_1 - 4)}$	$\rho(\theta_1 \mid \mathbf{x}) = \frac{2(2\sum_{j=1}^{n} x_{1j}^2 + b_1)^2}{(3n_1 + a_1 - 2)^2(3n_1 + a_1 - 4)}$
θ_2	$\rho(\theta_2 \mid \mathbf{x}) = \frac{8(\sum_{j=1}^{n} x_{2j}^2 + b_2)^2}{(3n_2 + 2a_2 - 2)^2(3n_2 + 2a_2 - 4)}$	$\rho(\theta_2 \mid \mathbf{x}) = \frac{2(2\sum_{j=1}^{n} x_{2j}^2 + b_2)^2}{(3n_2 + a_2 - 2)^2(3n_2 + a_2 - 4)}$
p	$\rho(p \mid \mathbf{x}) = \frac{(n_1+1)(n_2+1)}{(n+2)^2(n+3)}$	$\rho(p \mid \mathbf{x}) = \frac{(n_1+1)(n_2+1)}{(n+2)^2(n+3)}$

* Where IG & IC represents Inverted Gamma distribution and Inverted Chi-square distribution respectively.

9. Simulation Study

A thorough simulation study was conceded in order to investigate the performance of the Bayes estimators, impact of sample size and censoring rate in the fit of model. Sample sizes $n = 50, 100, 200, 300, 500$ were generated according to the criteria suggested by Krishna and Malik (2009) (for simple Maxwell distribution) from the two component mixture of Maxwell distribution with parameter θ_1, θ_2 and p such that $(\theta_1, \theta_2) \in \{(0.5, 0.8), (0.8, 1.5)\}$ and $p \in \{(0.25, 0.4)\}$ Probabilistic mixing was used to generate the mixture data. For each observation a random number k was generated from the uniform on $(0,1)$ distribution. If $k < p$, the observation was taken randomly from $\mathrm{f}_1(x)$ (the Maxwell distribution with parameter θ_1) and if $k > p$, the observation was taken randomly from $\mathrm{f}_2(x)$ (the Maxwell distribution with parameter θ_2).

Right censoring is carried out using a fixed censoring time t. All observations which are greater than T are stated as censored ones. Different fixed censoring times t are chosen to evaluate the impact of censoring rate on the estimates. The choice of the censoring time is made in such a way that the censoring rate in the resulting sample to be approximately 10% or 20%. For each of the different combination of parameters, sample size and censoring rate, different size of samples were generated using routine in Excel. In each case only failures are identified to be a member of either Subpopulation 1 or Subpopulation 2 of the mixture. For each of the 1000 samples, the Bayes estimates were computed using a routine in Mathematica and the results are presented in Table 5-12 given in appendix.The simulation study (appendix) provides us some interesting properties of the Bayes estimates. The properties of the estimates are highlighted in term of sample sizes, size of mixing proportion parameters, size of the component densities parameters, different loss functions and censoring rates. It is observed that due to censoring, the posterior risks of all three mixture parameters are reduced with an increase in sample size.

One can easily observe that the parameters of the component densities are generally over-estimated with a few exceptions in case of the second component. The extent of over-estimation is higher in case of the first component density parameter. On the other hand the estimates of the mixing proportion parameter are observed to be under-estimated with few values with increase in sample size. Another important point concerning about choice of loss function, SELF has less posterior risk than PLF, however underestimation some extent is prevented in PLF. If we make comparison between both uninformative (Uniform and Jeffreys) priors then due to less posterior risk the Jeffreys prior is more preferable than the uniform prior. Also comparison between informative priors, the IC (Inverted Chi-square) provides us less Bayes posterior risk than IG (Inverted Gamma) prior so IC prior is more suitable for this case. In over all comparison of informative priors on the behalf of less posterior risk are more preferable than noninformative priors and especially the IC informative prior is more preferable in present study.

10. Real Life Application

The burning velocity is the velocity of a laminar flame under stated conditions of composition, temperature, and pressure. The burning velocity is an important parameter which characterizes the inhibition efficiency of halogen-containing additive employed as flame retardants. The burning velocity decreases with increasing inhibitor concentration. It can be determined by analyzing the pressure–time profiles in the spherical vessel and were checked by direct observation of flame propagation. The data related to the burning velocity of different chemical materials available at the website (http://www.cheresources com/mists.pdf.). Data partition for mixture distribution is given in Appendix.

Table 13: BEs and PRs using UP and JP under SELF for real data.

Prior	**UP**			**JP**		
	E(θ_1|x)	**E(θ_2|x)**	**E(p|x)**	**E(θ_1|x)**	**E(θ_2|x)**	**E(p|x)**
p	**Censoring time= 65**					
0.25	1991.86 (293894)	1562.85 (54269.9)	0.266268 (0.00424)	1863.35 (239477)	1529.60 (51853.8)	0.266268 (0.00424)
0.45	1564.82 (90698.4)	1752.55 (97518.5)	0.466416 (0.00541)	1669.22 (118585)	1559.68 (65758.3)	0.400023 (0.00522)
0.50	1663.48 (92254.3)	1680.98 (99156.2)	0.511004 (0.00543)	1510.86 (81532.1)	1700.24 (88943.9)	0.466116 (0.00541)
0.60	1596.67 (65384.9)	1800.29 (166216)	0.644541 (0.00498)	1611.50 (83767.8)	1625.87 (89606.7)	0.511004 (0.00543)
	Censoring time= 70					
p	**E(θ_1|x)**	**E(θ_2|x)**	**E(p|x)**	**E(θ_1|x)**	**E(θ_2|x)**	**E(p|x)**
0.25	2094.13 (292340)	1790.12 (64730.4)	0.26501 (0.00389)	1970.94 (242796)	1755.36 (61011.3)	0.26501 (0.00389)
0.45	1707.35 (97186)	1857.41 (99998.1)	0.46920 (0.00498)	1654.00 (88244.0)	1806.52 (91935.5)	0.46920 (0.00498)
0.50	1839.58 (98075.4)	1740.16 (100943)	0.530533 (0.00498)	1789.18 (90164.9)	1685.78 (91675.8)	0.530533 (0.00498)
0.60	1694.12 (68317.4)	1958.26 (170418)	0.632726 (0.00465)	1655.61 (63755.5)	1878.33 (150116)	0.632726 (0.00464)

Table 14: BEs and PRs using UP and JP under PLF for real data.

Prior	UP			JP		
	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
p	Censoring time= 65					
0.25	2064.3159 (144.9119)	1580.1171 (34.5342)	0.274114 (0.015692)	1926.5384 (126.3768)	1561.4572 (33.7144)	0.274114 (0.015692)
0.45	1539.5369 (57.4339)	1780.1545 (55.2090)	0.472180 (0.011528)	1730.8227 (123.2054)	1580.6201 (41.8803)	0.406495 (0.012944)
0.50	1690.9820 (55.0039)	1710.2193 (58.4785)	0.516290 (0.010571)	1537.6053 (53.4906)	1726.1981 (51.9162)	0.471883 (0.011535)
0.60	1617.0158 (40.6915)	1845.8765 (90.1730)	0.648393 (0.007703)	1637.2843 (51.5687)	1653.1969 (54.6538)	0.516290 (0.010571)
	Censoring time= 70					
p	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
0.25	2162.8038 (137.3477)	1808.1095 (35.9790)	0.272250 (0.014481)	2031.6005 (121.3209)	1772.6534 (34.5868)	0.272250 (0.014481)
0.45	1735.5777 (56.4554)	1884.1364 (53.4528)	0.474477 (0.010554)	1680.4642 (52.9284)	1831.7887 (50.5375)	0.474477 (0.010554)
0.50	1866.0466 (52.9332)	1768.9262 (57.5323)	0.535206 (0.009346)	1814.2023 (50.0446)	1712.7551 (53.9502)	0.535206 (0.009346)
0.60	1714.1645 (40.0890)	2001.2996 (86.0793)	0.636390 (0.007328)	1674.7537 (38.2874)	1917.8737 (79.0874)	0.636390 (0.007328)

Table 15: BEs and PRs using Conjugate and IC Priors under SELF for real data.

Prior	Cojugate Prior			Inverted Chi-square Prior		
θ_1, θ_2	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
p	Censoring time = 65					
0.25	1725.29 (223854)	1522.54 (50263.3)	0.266268 (0.00424)	1722.54 (199456)	1510.43 (47835.4)	0.266268 (0.00424)
0.45	1601.53 (100698)	1552.55 (45547.1)	0.400023 (0.00522)	1612.21 (100458)	1559.68 (42753.5)	0.400023 (0.00522)
0.50	1502.43 (73251.7)	1680.98 (76156.2)	0.466116 (0.00541)	1494.83 (70534.6)	1700.24 (56973.9)	0.466116 (0.00541)
0.60	1521.33 (55367.2)	1698.34 (136254)	0.644541 (0.00498)	1511.33 (53757.8)	1702.87 (11674.3)	0.644541 (0.00498)
	Censoring time = 70					
p	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
0.25	1734.28 (192670)	1785.56 (53767.1)	0.26501 (0.00389)	1732.36 (172706)	1767.94 (51491.7)	0.26501 (0.00389)
0.45	1621.61 (88140)	1801.89 (89964.2)	0.46920 (0.00498)	1615.37 (88041.1)	1838.12 (81922.1)	0.46920 (0.00498)
0.50	1685.21 (78257.8)	1670.43 (85971)	0.530533 (0.00498)	1659.67 (78101.6)	1599.71 (61609.1)	0.530533 (0.00498)
0.60	1647.87 (45519.1)	1856.63 (140456)	0.632726 (0.00467)	1700.82 (39993.7)	1834.62 (140036)	0.632726 (0.004647)

Table 16: BEs and PRs using conjugate and IC priors under PLF for real data.

Prior	Conjugate Prior			Inverted Chi-square Prior		
θ_1,θ_2	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
p	**Censoring time = 65**					
0.25	1788.9884 (127.3968)	1538.9579 (32.8357)	0.274114 (0.015692)	1779.4943 (113.9086)	1526.1829 (31.5058)	0.274114 (0.015692)
0.45	1632.6654 (62.2708)	1567.1498 (29.1997)	0.406495 (0.012944)	1643.0700 (61.7200)	1573.3261 (27.2923)	0.406495 (0.012944)
0.50	1526.6100 (48.3662)	1703.4817 (45.0034)	0.471883 (0.011535)	1518.2395 (46.8191)	1716.9129 (33.3458)	0.471883 (0.011535)
0.60	1539.4194 (36.1788)	1737.9910 (79.3020)	0.648393 (0.007703)	1529.0115 (35.3630)	1706.2944 (6.8488)	0.648393 (0.004980)
	Censoring time = 70					
p	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
0.25	1788.9654 (109.3707)	1800.5531 (29.9863)	0.339130 (0.148241)	1781.5098 (98.2996)	1782.4431 (29.0063)	0.339130 (0.148241)
0.45	1648.5627 (53.9054)	1826.6833 (49.5865)	0.474477 (0.010554)	1642.3950 (54.0500)	1860.2707 (44.3015)	0.474477 (0.10554)
0.50	1708.2712 (46.1224)	1695.9680 (51.0759)	0.535206 (0.009346)	1683.0348 (46.7296)	1618.8518 (38.2836)	0.535206 (0.009346)
0.60	1661.6241 (27.5082)	1894.0779 (74.8957)	0.636406 (0.007359)	1712.5368 (23.4336)	1872.3959 (75.5519)	0.636406 (0.007359)

From Tables. 13-16, one can easily made comparison between results of uniform prior and Jeffery prior with their respective posterior risk which are given in parenthesis and can concludes that Jeffery prior has less variance (posterior risk) as compare to the uniform prior. Particularly when we use censoring time = 65 and censoring time = 70 our results are more precise. If we compare both informative priors, the IC prior has less posterior risk than the IG. In the same way the comparison between uninformative and informative priors, the IC provides less posterior risk so IC prior is more suitable prior.

10.2. Graphical presentation of Marginal posterior Densities

The graphs of the marginal posterior distributions for the parameters using Uniform and Jeffrey priors for real data set.

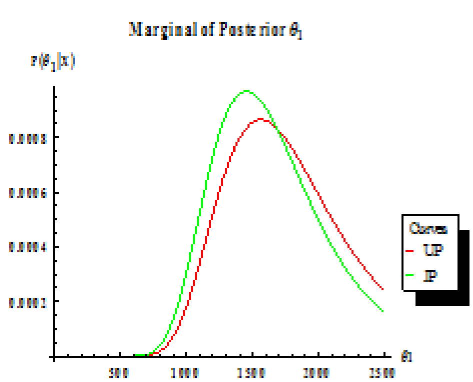

Fig.1

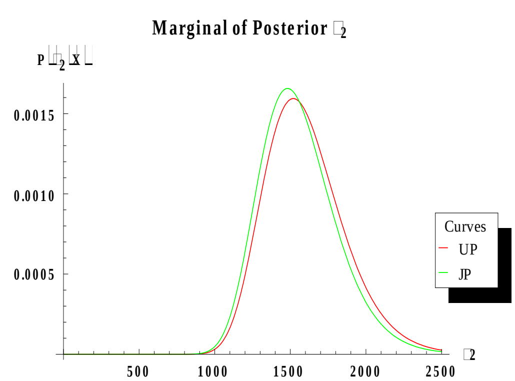

Fig.2

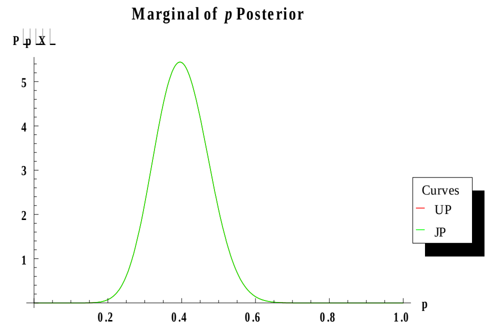

Fig.3

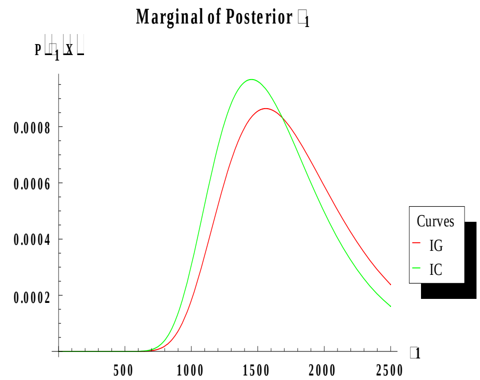

Fig.4

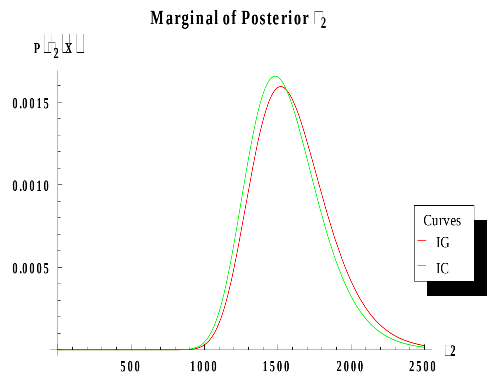

Fig.5

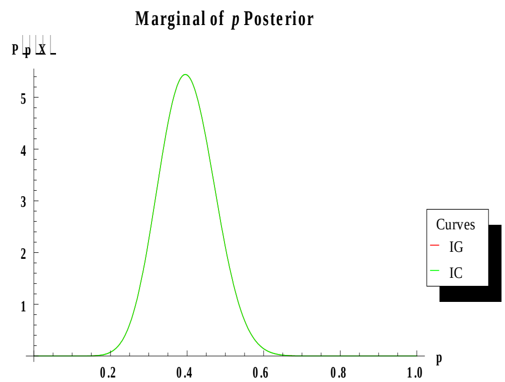

Fig.6

All graphs show similar pattern i.e. positive skewed, with miner difference.

11. Conclusion

The simulation study has displayed some fascinating properties of the Bayes estimates. The posterior risk of the parameter estimates seems to be fairly large in cases when the values of the parameters are large and fairly small for relatively smaller values of parameters. On the other hand in any case the posterior risk of estimates of both parameter θ_1 and θ_2 are reduced as the sample size increases. A further interesting observation about censoring the posterior risk of the estimates of θ_1 and θ_2 is that increasing or decreasing the proportion of a component in the mixture reduces (increasing) the analogous θ parameter's estimate.

The consequence of censoring on θ_1 is in the form of overestimation (underestimation) if θ_1 is less than θ_2 or θ_1 is greater than θ_2. To be more precise, larger degree of censoring time results in bigger sizes of over or underestimation. On the other hand the parameter p is either underestimated or overestimated depending upon the values of θ_1 and θ_2. To be more precise, p is over estimated or underestimated whenever $\theta_1 > \theta_2$ or $\theta_1 < \theta_2$.The level of this over or under estimation is directly proportional to amount of censoring rates and inversely proportional to the sample size. Also the level of over or under estimation is more intensive for larger parameter values of p. Further, the increase in sample size reduces the posterior risk of estimate of p.The increase in proportion of a component in the mixture does not guaranty the reduction in posterior risk of p. As the cut off sensor value gets infinity, the complete sample estimators and posterior risks are greatly simplified. Also posterior risks of the complete sample estimates are expected to be reduced further as these are clear from the effect of censoring time. All the Bayes estimates get more precise with the increase in sample size such that posterior risk using Jeffreys prior is less than the posterior risk of Uniform prior. In real life example, the estimates θ_1 and θ_2 are under estimated but much greater than the respective sample mean lifetime hours what is expected in censored samples.

The estimate of the mixing proportion parameter p is the same as that of the corresponding Mendenhall and Hader (1958) estimate. In case of informative priors, posterior risks using Inverted Chi-Square Prior are less than the posterior risks of Conjugate (Inverted Gamma) Prior. So on based on simulation study, we suggest at least 100 sample size for this type study. The posterior risks under SELF are less than the posterior risks under PLF; however underestimation is prevented in PLF. In future this work can be extended using mixture of truncated Maxwell distribution and taking beta prior for mixing component.

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Appendix

Following tables represents the Bayes estimates (BEs) and Posterior Risk (PRs) using different informative and noninformative priors under different loss functions.

Table 5: BEs and PRs using UP and JP under SELF when p=0.25.

Prior	UP			JP		
	E(θ_1 |x)	E(θ_2 |x)	E(p |x)	E(θ_1 |x)	E(θ_2 |x)	E(p |x)
n	Censoring time=0.6, θ_1 =0.5, θ_2 =0.8					
50	0.587884 (0.05963)	0.857126 (0.12243)	0.264686 (0.07263)	0.584119 (0.05204)	0.882172 (0.04864)	0.263437 (0.06641)
100	0.558953 (0.03237)	0.833803 (0.02504)	0.260129 (0.05242)	0.546332 (0.01505)	0.876655 (0.02164)	0.254718 (0.04512)
300	0.526199 (0.01334)	0.816405 (0.01435)	0.252168 (0.03032)	0.524204 (0.00311)	0.850066 (0.01304)	0.252085 (0.01561)
500	0.508690 (0.00819)	0.812348 (0.01247)	0.250275 (0.02951)	0.517223 (0.00309)	0.831584 (0.01025)	0.250416 (0.01193)
	Censoring time=1, θ_1 =0.8, θ_2 =1.5					
n	E(θ_1 |x)	E(θ_2 |x)	E(p |x)	E(θ_1 |x)	E(θ_2 |x)	E(p |x)
50	0.878715 (0.51476)	1.571415 (0.33957)	0.258421 (0.09998)	0.889619 (0.24782)	1.598889 (0.25253)	0.262068 (0.04831)
100	0.836967 (0.12406)	1.556432 (0.20691)	0.254329 (0.04495)	0.865673 (0.09242)	1.572284 (0.16195)	0.258753 (0.04139)
300	0.817587 (0.01393)	1.529194 (0.02165)	0.252198 (0.02922)	0.857527 (0.01301)	1.538463 (0.02142)	0.254169 (0.01024)
500	0.812845 (0.01056)	1.525878 (0.01912)	0.251636 (0.02042)	0.802448 (0.01036)	1.519759 (0.01129)	0.254169 (0.01724)

Table 6: BEs and PRs using UP and JP under SELF when p=0.40.

Prior	UP			JP		
	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
n	Censoring time=0.6, θ_1=0.5, θ_2=0.8					
50	0.578384 (0.05741)	0.890851 (0.09948)	0.501444 (0.08239)	0.581269 (0.03271)	0.877593 (0.06695)	0.477827 (0.06221)
100	0.564472 (0.03753)	0.851046 (0.04549)	0.498969 (0.04448)	0.577689 (0.03063)	0.869241 (0.03667)	0.467886 (0.01602)
300	0.520625 (0.01896)	0.826405 (0.03058)	0.422819 (0.02749)	0.535761 (0.01098)	0.824159 (0.01493)	0.415815 (0.00946)
500	0.517922 (0.01092)	0.790192 (0.02634)	0.413487 (0.01857)	0.526332 (0.00505)	0.813655 (0.01164)	0.251718 (0.00451)
	Censoring time=1, θ_1=0.8, θ_2=1.5					
n	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
50	0.893738 (0.10650)	1.597621 (1.56718)	0.468588 (0.01607)	0.889622 (0.10033)	1.582941 (1.36955)	0.476363 (0.01468)
100	0.851472 (0.08415)	1.576987 (0.54255)	0.462904 (0.00627)	0.854244 (0.05346)	1.557769 (0.43705)	0.444597 (0.00546)
300	0.835168 (0.02719)	1.532214 (0.04125)	0.429741 (0.00179)	0.812847 (0.02528)	1.521742 (0.02027)	0.417220 (0.00131)
500	0.812389 (0.01548)	1.515883 (0.01003)	0.426108 (0.00158)	0.811745 (0.01014)	1.515453 (0.00883)	0.413003 (0.00124)

Table 7: BEs and PRs using UP and JP under PLF when p=0.25.

Prior	Uniform Prior (UP)			Jeffreys Prior (JP)		
θ_1, θ_2	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
n	Censoring time=0.6, θ_2=0.5, θ_2=0.8					
50	0.636583 (0.097397)	0.925794 (0.137336)	0.377741 (0.113055)	0.627084 (0.085931)	0.909322 (0.054301)	0.368523 (0.210171)
100	0.587195 (0.056485)	0.848686 (0.029765)	0.346536 (0.172814)	0.559936 (0.027208)	0.888912 (0.024513)	0.331664 (0.153893)
300	0.538726 (0.025053)	0.825147 (0.017483)	0.306445 (0.108554)	0.527162 (0.005916)	0.857702 (0.015271)	0.281348 (0.058526)
500	0.516677 (0.015975)	0.819987 (0.015279)	0.303558 (0.106566)	0.520201 (0.005957)	0.837724 (0.012280)	0.273199 (0.045568)
θ_1, θ_2	Censoring time =1, θ_1=0.8, θ_2=1.5					
n	E(θ_1|x)	E(θ_2|x)	E(p|x)	E(θ_1|x)	E(θ_2|x)	E(p|x)
50	1.134416 (0.255701)	1.675982 (0.209133)	0.408364 (0.299887)	1.019432 (0.259626)	1.676000 (0.154222)	0.342037 (0.159939)
100	0.908060 (0.142187)	1.621539 (0.130215)	0.331109 (0.153560)	0.917502 (0.103658)	1.622668 (0.101369)	0.329155 (0.140804)
300	0.908879 (0.182585)	1.536256 (0.014125)	0.304670 (0.104944)	0.865079 (0.01515)	1.545409 (0.013891)	0.273572 (0.038806)
500	0.819315 (0.012939)	1.532130 (0.012505)	0.289380 (0.075488)	0.808877 (0.012859)	1.523469 (0.007419)	0.286080 (0.063822)

Table 8: BEs and PRs using UP and JP under PLF when p=0.40.

Prior	Uniform Prior (UP)			Jeffreys Prior (JP)		
θ_1,θ_2	E(θ_1|x)	E(θ_2|x)	E(p |x)	E(θ_1|x)	E(θ_2|x)	E(p |x)
n	Censoring time=0.6, θ_2 =0.5, θ_2 =0.8					
50	0.626049 (0.095331)	0.945037 (0.108373)	0.577785 (0.152683)	0.608756 (0.054974)	0.914942 (0.074699)	0.539007 (0.122360)
100	0.596790 (0.064636)	0.877365 (0.052638)	0.541710 (0.085483)	0.603618 (0.051858)	0.890084 (0.041686)	0.484703 (0.033635)
300	0.538526 (0.035802)	0.844704 (0.036598)	0.454165 (0.062692)	0.545912 (0.020302)	0.833167 (0.018017)	0.427039 (0.022447)
500	0.528359 (0.020874)	0.806687 (0.032989)	0.435364 (0.043753)	0.531108 (0.009551)	0.820777 (0.014243)	0.260522 (0.017608)
θ_1,θ_2	Censoring time =1, θ_1 =0.8, θ_2 =1.5					
n	E(θ_1|x)	E(θ_2|x)	E(p |x)	E(θ_1|x)	E(θ_2|x)	E(p |x)
50	0.951455 (0.115435)	2.029673 (0.864104)	0.485432 (0.033689)	0.944329 (0.109414)	1.968566 (0.771250)	0.491530 (0.030334)
100	0.899530 (0.096116)	1.740528 (0.327082)	0.469628 (0.013447)	0.884982 (0.061476)	1.692245 (0.268953)	0.450695 (0.012197)
300	0.851290 (0.032245)	1.545616 (0.02804)	0.431819 (0.004155)	0.828251 (0.030809)	1.528388 (0.013291)	0.418787 (0.003134)
500	0.821861 (0.018944)	1.519188 (0.006609)	0.427958 (0.003699)	0.817967 (0.012444)	1.518381 (0.005856)	0.414501 (0.002997)

Table 9: BEs and PRs using Conjugate and IC under SELF when p=0.25.

Prior	Conjugate Prior			Inverted Chi-square Prior		
θ_1,θ_2	E(θ_1|x)	E(θ_2|x)	E(p |x)	E(θ_1|x)	E(θ_2|x)	E(p |x)
n	Censoring time=0.6, θ_2 =0.5, θ_2 =0.8					
50	0.528732 (0.01067)	0.857452 (0.01679)	0.262496 (0.02214)	0.588339 (0.01036)	0.860826 (0.01635)	0.262496 (0.02128)
100	0.520666 (0.00792)	0.841602 (0.01185)	0.258923 (0.01384)	0.580629 (0.00771)	0.843672 (0.01161)	0.257231 (0.01297)
200	0.516292 (0.00586)	0.823471 (0.01008)	0.254391 (0.01141)	0.539769 (0.00579)	0.847651 (0.00982)	0.254982 (0.01098)
300	0.509098 (0.00419)	0.810703 (0.00346)	0.252018 (0.01050)	0.528035 (0.00416)	0.835989 (0.00325)	0.252436 (0.01012)
500	0.501227 (0.00224)	0.810351 (0.00117)	0.250188 (0.01011)	0.50122 (0.00209)	0.810324 (0.00112)	0.250416 (0.01007)
θ_1,θ_2	Censoring time =1, θ_1 =0.8, θ_2 =1.5					
n	E(θ_1|x)	E(θ_2|x)	E(p |x)	E(θ_1|x)	E(θ_2|x)	E(p |x)
50	0.859031 (0.17425)	1.576730 (0.20185)	0.262068 (0.01080)	0.890943 (0.17224)	1.584711 (0.15532)	0.261448 (0.01038)
100	0.858067 (0.06837)	1.547017 (0.10168)	0.256172 (0.00992)	0.861523 (0.06781)	1.581807 (0.10112)	0.253861 (0.00734)
200	0.830624 (0.01339)	1.546761 (0.01987)	0.255859 (0.00819)	0.855035 (0.01275)	1.538813 (0.01614)	0.253272 (0.00717)
300	0.814601 (0.00864)	1.519947 (0.01633)	0.252989 (0.00414)	0.824569 (0.00702)	1.513550 (0.01578)	0.250734 (0.00411)
500	0.801692 (0.00105)	1.502326 (0.01267)	0.250106 (0.00204)	0.801638 (0.00101)	1.502016 (0.01027)	0.252859 (0.01724)

Table 10: BEs and PRs using Conjugate and IC under SELF when p=0.40.

Prior	Conjugate Prior			Inverted Chi-square Prior		
θ_1, θ_2	E(θ_1|x)	E(θ_2 |x)	E(p |x)	E(θ_1|x)	E(θ_2 |x)	E(p |x)
n	Censoring time=0.6, θ_2 =0.5, θ_2 =0.8					
50	0.582698 (0.02868)	0.874514 (0.06402)	0.474856 (0.04159)	0.588339 (0.01036)	0.860826 (0.01635)	0.462496 (0.01128)
100	0.547051 (0.02363)	0.840387 (0.03342)	0.453181 (0.01267)	0.580629 (0.00771)	0.843672 (0.01161)	0.457231 (0.01097)
200	0.529334 (0.01057)	0.829302 (0.01358)	0.442489 (0.01098)	0.539769 (0.00579)	0.847651 (0.00982)	0.445982 (0.01032)
300	0.524493 (0.00856)	0.818559 (0.01183)	0.407992 (0.01012)	0.528035 (0.00416)	0.835989 (0.00325)	0.425243 (0.01007)
500	0.511454 (0.00276)	0.800339 (0.01071)	0.416376 (0.00857)	0.507132 (0.00505)	0.803645 (0.01164)	0.415171 (0.00217)
	Censoring time =1, θ_1 =0.8, θ_2 =1.5					
n	E(θ_1|x)	E(θ_2 |x)	E(p |x)	E(θ_1|x)	E(θ_2 |x)	E(p |x)
50	0.855675 (0.06239)	1.582241 (0.17269)	0.469661 (0.01208)	0.876131 (0.05658)	1.571380 (0.12164)	0.466735 (0.01107)
100	0.848314 (0.04133)	1.537606 (0.15608)	0.458176 (0.00528)	0.844649 (0.02628)	1.547638 (0.11701)	0.457405 (0.00501)
200	0.826327 (0.02187)	1.519541 (0.04234)	0.422854 (0.00159)	0.829453 (0.02085)	1.535807 (0.03611)	0.448474 (0.00321)
300	0.819506 (0.02016)	1.503132 (0.01428)	0.418382 (0.00122)	0.807223 (0.01965)	1.516132 (0.01383)	0.427825 (0.00116)
500	0.804383 (0.01011)	1.501827 (0.00765)	0.426108 (0.00119)	0.811745 (0.01008)	1.501745 (0.00694)	0.419403 (0.00116)

Table 11: BEs and PRs using Conjugate and IC under PLF when p=0.25.

Prior	Conjugate Prior			Inverted Chi-square Prior		
θ_1, θ_2	E(θ_1|x)	E(θ_2 |x)	E(p |x)	E(θ_1|x)	E(θ_2 |x)	E(p |x)
n	Censoring time=0.6, θ_2 =0.5, θ_2 =0.8					
50	0.538728 (0.019991)	0.867187 (0.019471)	0.301735 (0.078478)	0.597078 (0.017479)	0.870271 (0.018889)	0.3000307 (0.075621)
100	0.528217 (0.015102)	0.848613 (0.014022)	0.0284396 (0.050946)	0.587231 (0.013204)	0.850525 (0.0130706)	0.281314 (0.048167)
200	0.521936 (0.011288)	0.829569 (0.012196)	0.275907 (0.043032)	0.545106 (0.010674)	0.853424 (0.011546)	0.275673 (0.041383)
300	0.513197 (0.008198)	0.812834 (0.004262)	0.272053 (0.040071)	0.531959 (0.007849)	0.837930 (0.003883)	0.271742 (0.038613)
500	0.503456 (0.004459)	0.811072 (0.001443)	0.269637 (0.038898)	0.503300 (0.004161)	0.811015 (0.001381)	0.269657 (0.038483)
θ_1, θ_2	Censoring time =1, θ_1 =0.8, θ_2 =1.5					
n	E(θ_1|x)	E(θ_2 |x)	E(p |x)	E(θ_1|x)	E(θ_2 |x)	E(p |x)
50	0.955083 (0.192105)	1.639490 (0.15520)	0.281921 (0.039707)	0.982863 (0.183840)	1.632982 (0.096541)	0.280598 (0.038299)
100	0.897022 (0.077910)	1.579538 (0.065043)	0.274853 (0.037362)	0.900018 (0.076989)	1.613454 (0.063294)	0.267928 (0.028134)
200	0.838645 (0.016043)	1.553171 (0.012819)	0.271392 (0.031067)	0.862459 (0.014847)	1.544048 (0.010471)	0.267052 (0.027560)
300	0.819887 (0.010572)	1.525309 (0.010725)	0.261043 (0.016108)	0.828815 (0.008492)	1.518754 (0.010408)	0.258800 (0.016132)
500	0.802346 (0.001309)	1.506537 (0.008422)	0.288050 (0.075889)	0.802268 (0.001259)	1.505431 (0.006830)	0.284917 (0.064116)

Table 12: BEs and PRs using Conjugate and IC priors under PLF when p=0.40.

Prior	Conjugate Prior			Inverted Chi-square Prior		
θ_1, θ_2	E(θ_1 |x)	E(θ_2 |x)	E(p |x)	E(θ_1 |x)	E(θ_2 |x)	E(p |x)
n	**Censoring time=0.6, θ_2 =0.5, θ_2 =0.8**					
50	0.606808 (0.048222)	0.910382 (0.071735)	0.516796 (0.083880)	0.597078 (0.017479)	0.870271 (0.018890)	0.474534 (0.024076)
100	0.568238 (0.042375)	0.860041 (0.039308)	0.466951 (0.027539)	0.587231 (0.013204)	0.850525 (0.013706)	0.469074 (0.023685)
200	0.539226 (0.019784)	0.837449 (0.016295)	0.454727 (0.024476)	0.545106 (0.010674)	0.853424 (0.011546)	0.457406 (0.022847)
300	0.532591 (0.016195)	0.825753 (0.014389)	0.420211 (0.024438)	0.531959 (0.007849)	0.837930 (0.003883)	0.436923 (0.023359)
500	0.514145 (0.005382)	0.807002 (0.013326)	0.426543 (0.020334)	0.512087 (0.009909)	0.810855 (0.014419)	0.417776 (0.005210)
	Censoring time =1, θ_1 =0.8, θ_2 =1.5					
n	E(θ_1 |x)	E(θ_2 |x)	E(p |x)	E(θ_1 |x)	E(θ_2 |x)	E(p |x)
50	0.891386 (0.071423)	1.635902 (0.107323)	0.482350 (0.025378)	0.907847 (0.063431)	1.609619 (0.076479)	0.478447 (0.023424)
100	0.872334 (0.048040)	1.587549 (0.136016)	0.463902 (0.011452)	0.860065 (0.030832)	1.584990 (0.074704)	0.462849 (0.010888)
200	0.839456 (0.026258)	1.533409 (0.027737)	0.424730 (0.003752)	0.841928 (0.024949)	1.547518 (0.023423)	0.452039 (0.007129)
300	0.831715 (0.024418)	1.507875 (0.009485)	0.419837 (0.002911)	0.819304 (0.024162)	1.520686 (0.009108)	0.429178 (0.002707)
500	0.810643 (0.012520)	1.504372 (0.005089)	0.427502 (0.002788)	0.817930 (0.012370)	1.504054 (0.004618)	0.420784 (0.002761)

Real data set for mixture of Maxwell model as following:
Censoring time= 65.

$$p=0.25, n_1=14, n_2=42,\ \ r_1=11, r_2=32,\ \sum_{j=1}^{r_1} x_{1j}^{\ 2}=28882,\ \sum_{j=1}^{r_2} x_{2j}^{\ 2}=71891, D_1=0.12070, D_2=0.114325.$$

$$p=0.45, n_1=25, n_2=31,\ \ r_1=20, r_2=23,\ \sum_{j=1}^{r_1} x_{1j}^{\ 2}=43815,\ \sum_{j=1}^{r_2} x_{2j}^{\ 2}=56958, D_1=0.117697, D_2=0.11454.$$

$$p=0.50, n_1=28, n_2=28,\ \ r_1=22, r_2=21,\ \sum_{j=1}^{r_1} x_{1j}^{\ 2}=51568,\ \sum_{j=1}^{r_2} x_{2j}^{\ 2}=49589, D_1=0.116165, D_2=0.114826.$$

$$p=0.60, n_1=26, n_2=14,\ \ r_1=28, r_2=15,\ \sum_{j=1}^{r_1} x_{1j}^{\ 2}=63867,\ \sum_{j=1}^{r_2} x_{2j}^{\ 2}=36906, D_1=0.114834, D_2=0.116153.$$

Censoring time= 70.

$$p=0.25, n_1=14, n_2=42,\ \ r_1=12, r_2=35,\ \sum_{j=1}^{r_1} x_{1j}^{\ 2}=33506,\ \sum_{j=1}^{r_2} x_{2j}^{\ 2}=90401, D_1=0.12202, D_2=0.114561.$$

$$p=0.45, n_1=25, n_2=31,\ \ r_1=22, r_2=25,\ \sum_{j=1}^{r_1} x_{1j}^{\ 2}=52928,\ \sum_{j=1}^{r_2} x_{2j}^{\ 2}=65938, D_1=0.118492, D_2=0.114784.$$

$$p=0.50, n_1=28, n_2=28,\ \ r_1=25, r_2=22,\ \sum_{j=1}^{r_1} x_{1j}^{\ 2}=65305,\ \sum_{j=1}^{r_2} x_{2j}^{\ 2}=53945, D_1=0.116703, D_2=0.115143.$$

$$p=0.60, n_1=26, n_2=14,\ \ r_1=30, r_2=17,\ \sum_{j=1}^{r_1} x_{1j}^{\ 2}=72847,\ \sum_{j=1}^{r_2} x_{2j}^{\ 2}=46019, D_1=0.115153, D_2=0.116689.$$