Impulsive Synchronization of Lorenz Systems by the Lyapunov-Razumikhin Method

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Abstract

This paper studies impulsive synchronization of two Lorenz systems. An impulsive control scheme which considers transmission delay in chaos-based secure communication is presented. Sufficient conditions for the globally exponential stabilization of impulsive synchronization of two Lorenz systems are established by employing the Lyapunov-Razumikhin method. Upper bounds of the length and strength of the impulsive control are presented. Simulation results are discussed to illustrate the theorems.

Keywords: Impulsive synchronization; Lyapunov-Razumikhin technique; Lorenz system; Global exponential stability

1. Introduction

Recently the control and stabilization of chaotic systems have attracted lots of attention due to their applications in secure communications ([2] - [7], [17] - [23]). A variety of approaches including adaptive control, feedback control, OGY method, predictive Poincare control and impulsive control were presented in research articles ([23]). Compared to other methods, impulsive control has an advantage in dealing with systems which cannot endure continuous disturbance ([20]). The approach, which allows the stabilization of the impulsive synchronization using small impulses, is based on digital control devices to generate control impulses in discrete moments.

Early development of this method has been focused on obtaining sufficient conditions to asymptotically stabilize the impulsive synchronization of the error dynamics between driving and response chaotic systems which was modeled by impulsive ordinary differential equations ([17] - [23]). Recently, transmission delay and sampling delay have been considered in the models and sufficient conditions on achieving equi-attractivity properties of the impulsive synchronization were given in [6]. Theorems dealing with robustness and parameter mismatches in impulsive control schemes were presented in [2] and [7]. However, there are very few publications on the global exponential stability of the impulsive synchronization of chaotic systems. The global exponential stability of the error dynamics between the driving and response systems will guarantee a fast impulsive synchronization regardless of big differences between the initial values of the two chaotic systems. In this paper, we investigate an impulsive control scheme which considers transmission delay in secure communications. Based on the Lyapunov-Razumikhin method, sufficient conditions for the global exponential stabilization of impulsive synchronization of two Lorenz systems are presented. Upper bounds of the length and strength of the impulsive control are given. Some simulation results are discussed to illustrate our results.

2. Global Exponential Stability of Impulsive Synchronization

In this section, we shall first present known results for the stability of a general impulsive delay differential system from [16, 18]. Then we study the impulsive control of Lorenz systems by applying the theory.

Let $\Re^n$ denote the n-dimensional real space and $\Re_+ = [0, \infty)$, and let $\aleph$ denote the set of positive integers, i.e. $\aleph = \{1, 2, \ldots\}$. Let $\lambda_{\max}(Q)$ (or $\lambda_{\min}(Q)$) denote the maximum (or minimum) eigenvalue of a symmetric matrix Q, and $\|A\|$ the norm of matrix A induced by the Euclidean vector norm, i.e., $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$.

Given a constant $\tau > 0$, we equip the space $PC([-\tau,0],\Re^n)$ with the norm $\|\cdot\|_\tau$ defined by $\|\psi\|_\tau = \sup_{-\tau \le s \le 0} \|\psi(s)\|$ for any $\psi \in PC([-\tau,0],\Re^n)$, where

$$PC([-\tau,0],\Re^n) = \{\psi : [-\tau,0] \to \Re^n \mid \psi(t) = \psi(t^+), \forall t \in [-\tau,0); \psi(t^-) \text{ exists in } \Re^n,$$
$$\forall t \in (-\tau,0], \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number of points } t \in (-\tau,0]\}.$$

Consider the general nonlinear impulsive system with time delay

$$\begin{cases} x'(t) = F(t, x_t), & t \ne t_k, \\ \Delta x(t_k) = I_k(t_k, x_{t_k^-}), & k \in \aleph, \\ x_{t_0} = \phi, \end{cases} \tag{2.1}$$

where F, $I_k : \Re_+ \times PC([-\tau,0],\Re^n) \to \Re^n$, $\phi \in PC([-\tau,0], \Re^n)$, and $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, with $t_k \to \infty$ as $k \to \infty$. $\Delta x(t) = x(t) - x(t^-)$ and $x_t, x_{t^-} \in PC([-\tau,0],\Re^n)$ are defined by $x_t(s) = x(t+s), x_{t^-}(s) = x(t^- + s)$ for $-\tau \le s \le 0$, respectively.

Definition 2.1 *A function $V : \Re_+ \times \Re^n \to \Re_+$ is said to belong to the class υ_0 if*

i) V is continuous in each of the sets $[t_{k-1}, t_k] \times \Re^n$ and for each $x \in \Re^n$, $t \in [t_{k-1}, t_k), k \in \aleph$, $\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-, x)$ exists; and

ii) V(t,x) is locally Lipschitzian in all $x \in \Re^n$ and $V(t,0) \equiv 0$ for all $t \ge t_0$.

Definition 2.2 *Given a function $V : \Re_+ \times \Re^n \to \Re_+$, the upper right-hand derivative of V with respect to system (2.1) is defined by*

$$D^+V(t,\psi(0)) = \limsup_{h\to 0^+} \frac{1}{h}\left[V(t+h,\psi(0)+hF(t,\psi)) - V(t,\psi(0))\right],$$

For all $(t,\psi) \in \Re_+ \times PC([-\tau,0],\Re^n)$.

Definition 2.3 *The trivial solution of system (2.1) is said to be globally exponentially stable, if there exist some constants $\alpha > 0$ and $M \ge 1$ such that for any initial data $x_{t_0} = \phi$,*

$$\|x(t,t_0,\phi)\| \le M\|\phi\|_\tau e^{-\alpha(t-t_0)}, \qquad t \ge t_0,$$

where $(t_0,\phi) \in \Re_+ \times PC([-\tau,0],\Re^n)$, α is called the convergence rate.

Lemma 2.1[16] *Assume that there exist a function $V \in \upsilon_0$, constants $p, c, c_1, c_2 > 0$ and $\alpha > \tau, \lambda > c$ such that*

(i) $c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$, for any $t \in \Re_+$ and $x \in \Re^n$;

(ii) $D^+V(t,\varphi(0)) \le cV(t,\varphi(0))$, for all $t \in [t_{k-1}, t_k)$, $k \in \aleph$,
whenever $qV(t,\varphi(0)) \ge V(t+s,\varphi(s))$ for $s \in [-\tau,0]$, where $q \ge e^{2\lambda\alpha}$ is a constant;

(iii) $V(t_k, \varphi(0) + I_k(t_k,\varphi)) \le d_k V(t_k^-, \varphi(0))$, where $d_k > 0, \forall k \in \aleph$ are constants;

(iv) $\tau \le t_k - t_{k-1} \le \alpha$ and $\ln(d_k) + \lambda\alpha \le -\lambda(t_k - t_{k-1})$.

Then the trivial solution of the impulsive system (2.1) is globally exponentially stable and the convergence rate is $\frac{\lambda}{p}$.

Lemma 2.2[18] *Given a positive definite matrix* $P \in \Re^{n\times n}$, *any symmetric matrix* $Q \in \Re^{n\times n}$ *and* $x \in \Re^{n}$, *then*

$$\lambda_{\min}(P^{-1}Q)x^T Px \le x^T Qx \le \lambda_{\max}(P^{-1}Q)x^T Px. \qquad (2.2)$$

Next, we shall consider the stabilization of the error dynamical system of two Lorenz systems using impulsive control.

A famous example of chaotic system, the Lorenz system [14], is given by

$$\begin{cases} x' = -\sigma x + \sigma y, \\ y' = rx - y - xz, \\ z' = xy - bz, \end{cases} \qquad (2.3)$$

where σ, r, and b are real positive numbers. This system can be rewritten in the following vector form

$$X'(t) = AX(t) + \Phi_1(X(t)) + \Phi_2(X(t)), \qquad (2.4)$$

where $X^T(t) = (x, y, z)$ and

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ \mathrm{r} & -1 & 0 \\ 0 & 0 & \mathrm{-b} \end{bmatrix}, \qquad \Phi_1(X) = \begin{bmatrix} 0 \\ -\mathrm{xz} \\ 0 \end{bmatrix}, \qquad \Phi_2(X) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{xy} \end{bmatrix}. \qquad (2.5)$$

In the cryptosystem proposed by Khadra et al. [5,6], the driving system at the transmitter end is represented by equation (2.4), while the response system U with $U(t) = (u(t), v(t), w(t))^T$ at the receiver end is given by

$$\begin{cases} U'(t) = AU(t) + \Phi_1(U(t)) + \Phi_1(X(t-\tilde{\tau})) - \Phi_1(U(t-\tilde{\tau})) + \Phi_2(U(t)), & \mathrm{t} \ne t_k, \\ \Delta U(t) = -B_k e(t), & \mathrm{t} = t_k,\ \mathrm{k} \in \aleph, \end{cases} \qquad (2.6)$$

where $\Delta U(t_k) = U(t_k^+) - U(t_k^-) = U(t_k) - U(t_k^-)$ and B_k $(k \in \aleph)$ are 3×3 constant matrices describing the linear nature of the driving impulses, $\tilde{\tau}^T = (\tau_x, \tau_y, \tau_z)$ represents the transmission delay and $\tau = \max\{\tau_x, \tau_y, \tau_z\}$ is a positive constant, and $e(t) = X(t) - U(t) = (e_x(t), e_y(t), e_z(t))^T$ denotes the error dynamics.

The error system of the impulsive synchronization is obtained by subtracting (2.6) from (2.4):

$$\begin{cases} e'(t) = Ae(t) + \Psi_1(X(t), U(t)) + \Psi_1(X(t-\tilde{\tau}), U(t-\tilde{\tau})) + \Psi_2(X(t), U(t)), & \mathrm{t} \ne t_k, \\ \Delta e(t) = B_k e(t), & \mathrm{t} = t_k,\ \mathrm{k} \in \aleph, \end{cases} \qquad (2.7)$$

where $\Psi_1(X(t), U(t)) = \Phi_1(X(t)) - \Phi_1(U(t))$ and $\Psi_2(X(t), U(t)) = \Phi_2(X(t)) - \Phi_2(U(t))$.
Since chaotic signals are bounded, there exist positive constants L_1 and L_2 such that $\|\Psi_1(X, U)\| \le L_1\|X - U\|$ and $\|\Psi_2(X, U)\| \le L_2\|X - U\|$ ([6, 21]). For system (2.7), we have $L_1 = L_2 = 2\tilde{M}$ with $\tilde{M} = \max\{|x|, |y|, |z|\}$.

We shall analyze the dynamics of system (2.7) and find the conditions under which the global exponential stability property may be achieved.

Theorem 2.1 *Let $Q = PA + A^T P$, where P is an $n \times n$ symmetric and positive definite matrix. Let the impulses be equidistant from each other and separated by interval Δ. If there exists constants $\alpha > \tau$ and $\lambda > c$ with $q \geq e^{2\lambda\alpha}$ where*

$$c = \lambda_{\max}(P^{-1}Q) + 2\left[L_1\left(1 + \sqrt{\frac{q\lambda_{\max}(P)}{\lambda_{\min}(P)}}\right) + L_2\right]\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$$

such that

$$\tau \leq \Delta \leq \alpha, \qquad \lambda_{\max}(Q^{-1}(I + B_k)^T P(I + B_k)) < e^{-\lambda(\alpha+\Delta)}, \qquad (2.8)$$

where I is the identity matrix. Then the impulsive synchronization between two identical Lorenz systems is globally exponentially stable and the convergence rate is $\frac{1}{2}\lambda$.

Proof. Construct a Lyapunov function $V(t, e(t)) = e^T(t)Pe(t)$. Then

$$\lambda_{min}(P)e^2(t) \leq V(t, e(t)) \leq \lambda_{max}(P)e^2(t),$$

condition (i) of Lemma 2.1 is satisfied with $c_1 = \lambda_{min}(P)$ and $c_2 = \lambda_{max}(P)$. We know that Q is symmetric since P is symmetric.

For $t \neq t_k$, calculate the upper right-hand derivative of V along system (2.7):

$$\begin{aligned} D^+V(t, e(t)) &= (e'(t))^T Pe(t) + e^T(t)Pe'(t) \\ &= e^T(t)(A^T P + PA)e(t) + 2\Psi_1^T(X(t), U(t))Pe(t) \\ &\quad + 2\Psi_1^T(X(t-\tilde{\tau}), U(t-\tilde{\tau}))Pe(t) + 2\Psi_2^T(X(t), U(t))Pe(t) \\ &\leq e^T(t)Qe(t) + 2L_1\|P\|e^2(t) + 2L_1\|e(t-\tilde{\tau})\|\,\|P\|\,\|e(t)\| \\ &\quad\quad +2L_2\|P\|e^2(t), \end{aligned}$$

since $\|\Psi_m(X, U)\| \leq L_m\|X(t) - U(t)\| = L_m\|e(t)\|$ for $m = 1,2$. Let $q \geq e^{2\lambda\alpha}$, whenever $qV(t, \varphi(0)) \geq V(t + s, \varphi(s))$, i.e., $qe^T(t)Pe(t) \geq e^T(t + s)Pe(t + s)$ for $s \in [-\tau, 0]$, we have

$$\|e(t + s)\| \leq \sqrt{\frac{q\lambda_{max}(P)}{\lambda_{min}(P)}}\|e(t)\|$$

for $s \in [-\tau, 0]$, and then by Lemma 2.2, for $t \neq t_k$ with $k \in \aleph$,

$$\begin{aligned} D^+V(t, e(t)) &= \lambda_{max}(P^{-1}Q)V(t, e(t)) + 2L_1\lambda_{max}(P)e^2(t) \\ &\quad\quad +2L_1\lambda_{max}(P)\|e(t)\|\,\|e(t-\tilde{\tau})\| + 2L_2\lambda_{max}(P)e^2(t) \\ &\leq \lambda_{max}(P^{-1}Q)V(t, e(t)) \\ &\quad + 2\lambda_{\max}(P)\left[L_1\left(1 + \sqrt{\frac{q\lambda_{\max}(P)}{\lambda_{\min}(P)}}\right) + L_2\right]e^2(t) \\ &\leq cV(t, e(t)), \end{aligned}$$

where $c = \lambda_{\max}(P^{-1}Q) + 2\left[L_1\left(1 + \sqrt{\frac{q\lambda_{\max}(P)}{\lambda_{\min}(P)}}\right) + L_2\right]\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$. Thus the condition (*ii*) of Lemma 2.1 holds. Furthermore, since $(I + B_k)^T P(I + B_k)$ is symmetric, by Lemma 2.2, we have

$$\begin{aligned} V(t_k, e(t_k)) &= e^T(t_k)Pe(t_k) \\ &= e^T(t_k^-)(I + B_k)^T P(I + B_k)e(t_k^-) \\ &\leq \lambda_{max}(Q^{-1}(I + B_k)^T P(I + B_k))V(t_k^-, e(t_k^-)), \end{aligned}$$

which implies that the condition (*iii*) of Lemma 2.1 holds. Notice that the assumption (2.8) yields the condition (*iv*) of Lemma 2.1, hence it follows from Lemma 2.1 that the trivial solution of (2.7) is globally exponentially stable.

The following result can be obtained by substituting $P = I$ in Theorem 2.1.

Corollary 2.1 *Let* $Q = A + A^T$ *and the impulses be equidistant from each other and separated by interval* Δ. *If there exists constants* $\alpha > \tau$ *and* $\lambda > c \triangleq \lambda_{max}(Q) + 2[L_1(1+\sqrt{q}) + L_2]$ *with* $q \geq e^{2\lambda\alpha}$ *such that*

$$\tau \leq \Delta \leq \alpha, \|I + B_k\| < \sqrt{\frac{e^{-\lambda(\alpha+\Delta)}}{\lambda_{max}(Q^{-1})}} \tag{2.9}$$

where I is the identity matrix. Then the impulsive synchronization between two identical Lorenz systems is globally exponentially stable and the convergence rate is $\frac{1}{2}\lambda$.

3. Numerical Simulation Results

In this section, the parameters of the Lorenz system are chosen as σ =10, r = 28, and $b = \frac{8}{3}$. The graph of this Lorenz system is shown in Figure 1 with initial condition $(x_0, y_0, z_0)^T = (-0.12, 0.25, -0.005)^T$. We notice that $\tilde{M} = \max\{x, y, z\} = 50$.

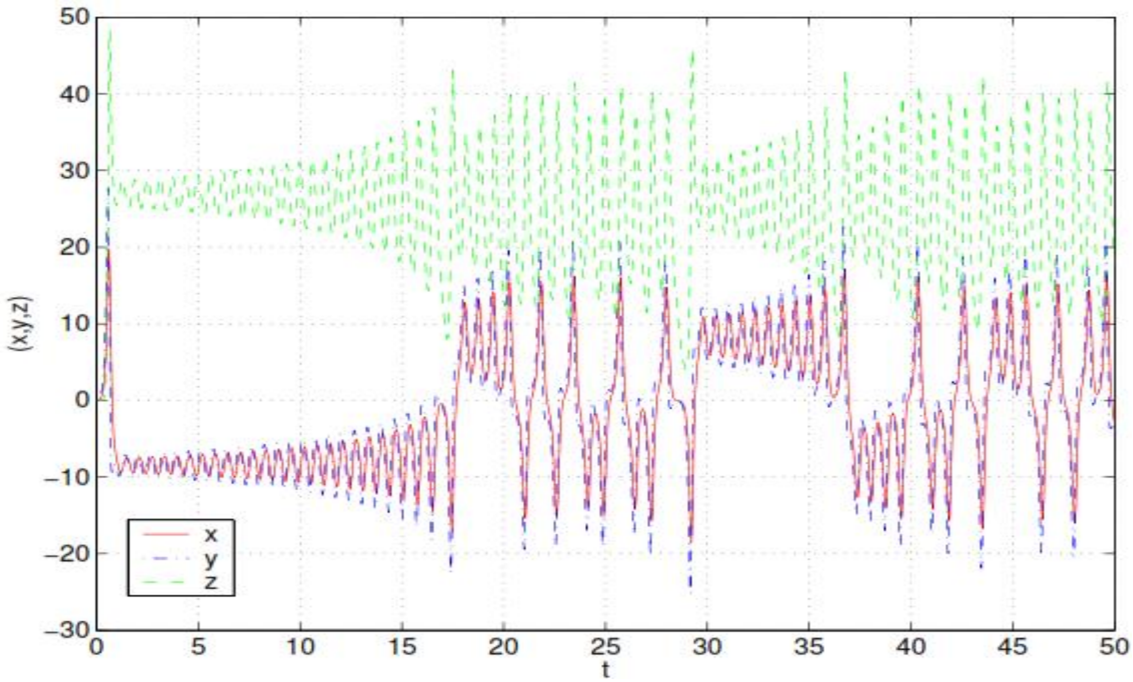

Figure 1. Lorenz system.

3.1 Achieved Impulsive Synchronization

In this simulation, we discuss impulsive synchronization with strong couplings. Choose λ = 742, q = 2.45, t_k = 0.0005k, τ = Δ = 0.0005, α = 0.0006 and B_k = -0.99I. Then the conditions of Corollary 2.1 are satisfied. The numerical simulation of e with initial conditions $X(0)=(1.1, -1, 0.8)^T$ and $U(0)=(0.9, -1.2, 0.7)^T$ is given in Figure 2.

From Figure 2, we see that the impulsive synchronization of the driving and response systems was achieved in a short time regardless of the big mismatches between the initial conditions $X(0)$ and $U(0)$.

The numerical simulations of the driving and response systems are given in Figures 3 and 4 with initial conditions $X(0) = (1.1, -1, 0.8)^T$ and $U(0) = (0.9, -1.2, 0.7)^T$, respectively.

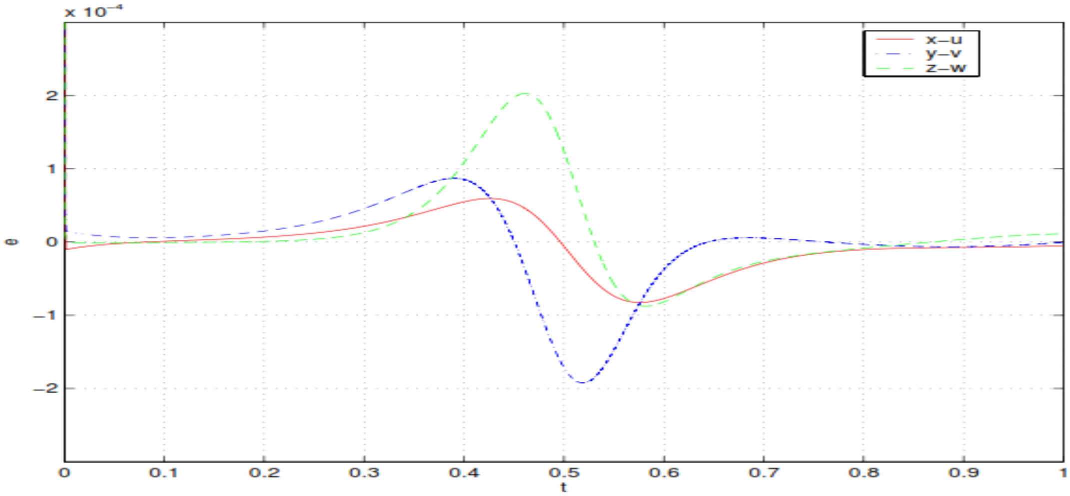

Figure 2. Impulsively synchronized error dynamical system *e*.

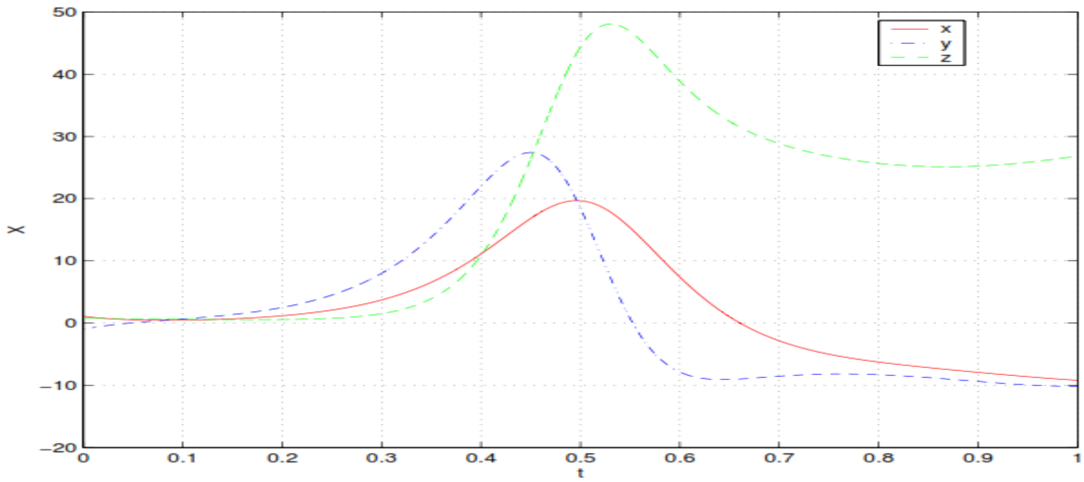

Figure 3: Driving system *X*.

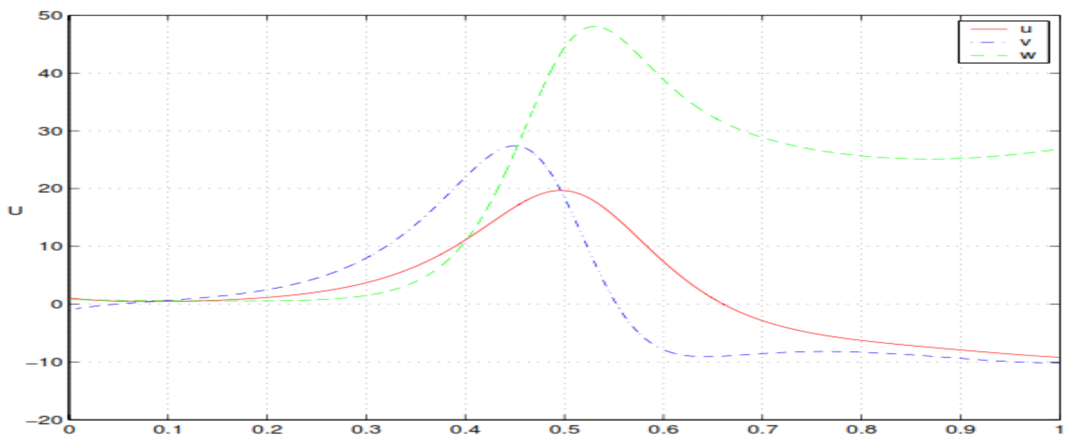

Figure 4: Response system *U*.

3.2. Failed Impulsive Synchronization

If we change the linear impulsive matrix to B_k = diag{-1.5,0.5,-0.8} and keep the values of other parameters the same as in simulation 3.1, then the synchronization cannot be achieved. The numerical simulations of the error dynamics, driving system, and response system are given in Figures 5, 6 and 7, respectively. From Figure 5, we notice that the trivial solution of the error dynamics became increasingly large in a short time because of the big mismatches between the initial conditions of the two chaotic systems, the presence of the transmission delay, and the not-strong-enough impulsive control.

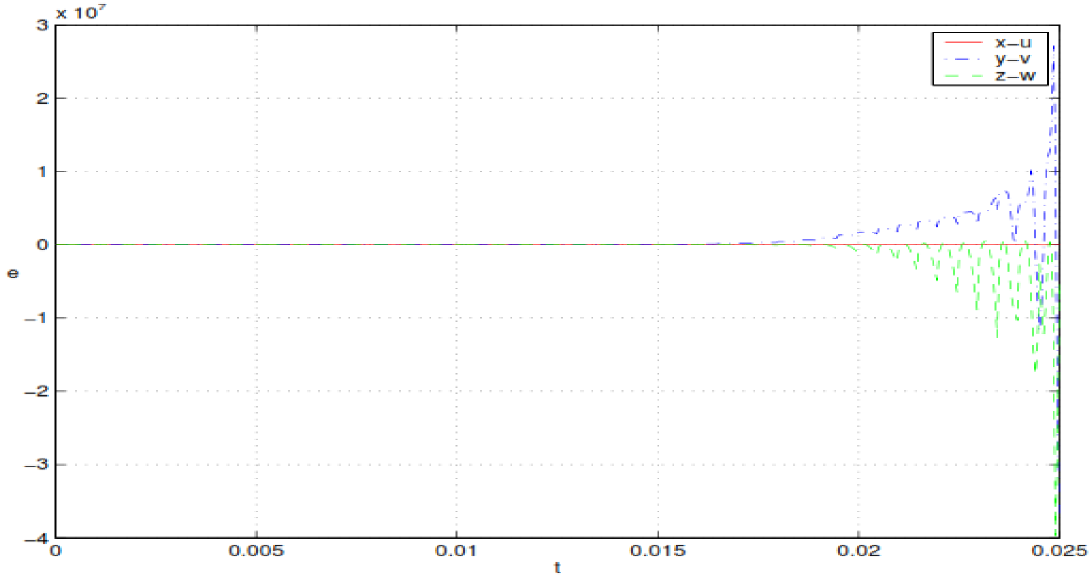

Figure 5: Unstable error dynamical system *e*.

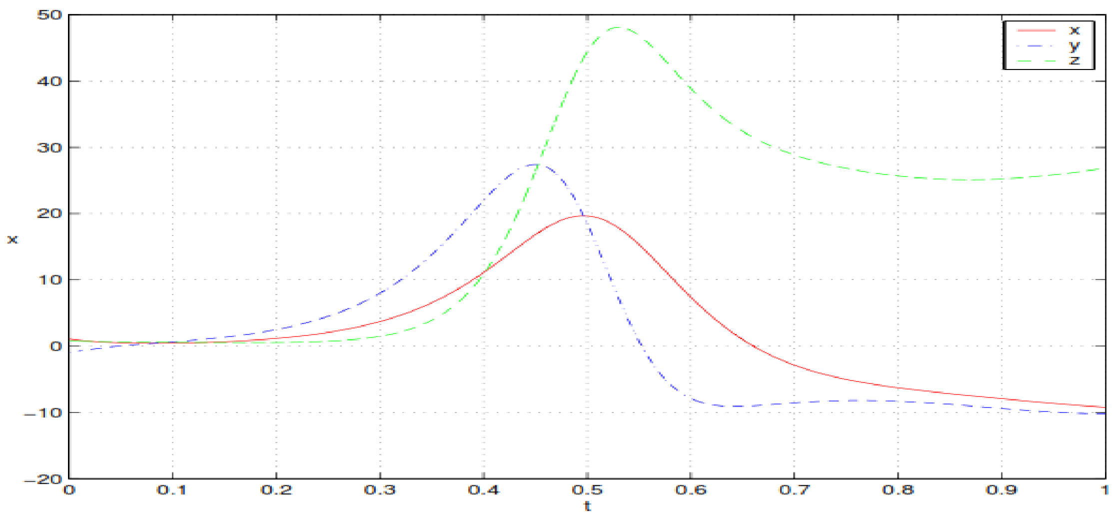

Figure 6: Driving system *X*.

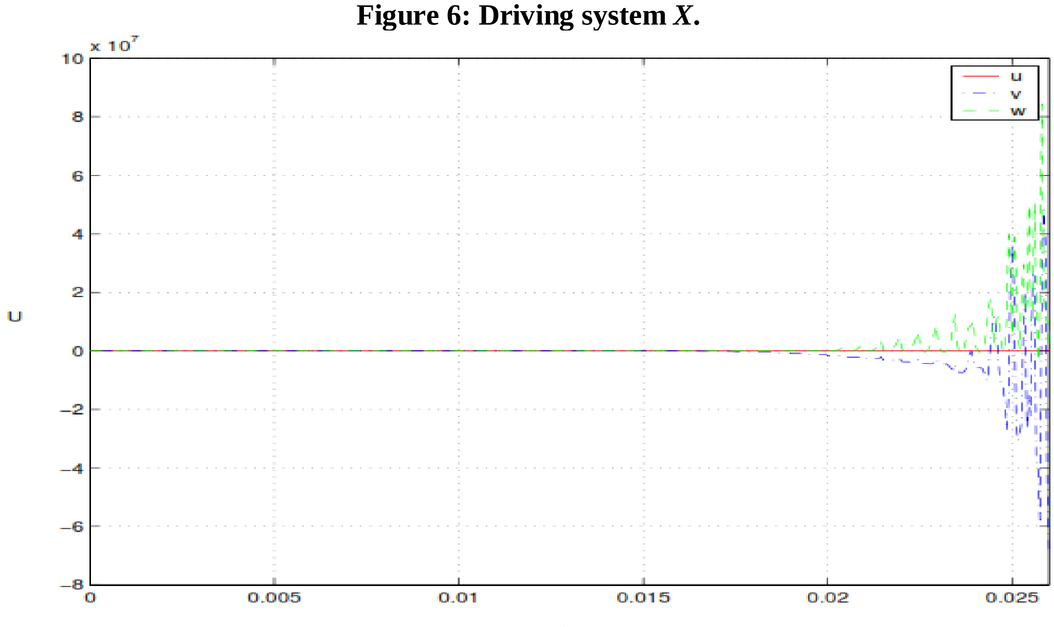

Figure 7: Response system *U*.

4. Conclusions

In this paper, we investigated an impulsive control scheme of two Lorenz systems. Transmission delay is considered in our model. Sufficient conditions on globally exponential stabilization of impulsive synchronization were presented, which would guarantee fast synchronization of two chaotic systems with large mismatches in initial values of the two systems. Upper bounds of the length and strength of impulsive control are also given.

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