Some Estimates below the Modulus of Integrals of Real Polynomials in the Complex Plane

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Abstract

In this paper, we make some estimates below the modulus of some integrals in the complex plane using division a polynomial into other.

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1. Introduction

Here we extend the learning estimates of the modulus of some complex integrals. Inequalities which are obtained are below the modulus of some integrals in the complex plane. Our proof here is completely different from the proof in [8]. Now we use a division between polynomials. Theorem5 is new. It is relevant to the case n=4. The general case is unknown.

The integral function is a polynomial $f(x) = x(x + a_1)(x + a_2) \dots (x + a_n)$, where $a_k \geq 0$. The general conjecture are the inequalities

$$\left| \int_0^{e^{i\varphi}} x \prod_{k=1}^{n} (x + a_k) dx \right| \geq \frac{1}{n+2} ,$$

which are proved for $n = 1,2,3,4,5$, $a_k \geq 0$, $a_k \in \mathbb{R}$. We can see the results of Theorem1 in [8-9]. Such ones of Theorem2 and Theorem3 could be seen in [10]. We present the proof of Theorem4, because of better understanding of Theorem5. Here in Theorem5 we consider the condition $0 \leq a \leq b \leq c \leq d \leq 1$. But according to general conjecture we can replace this condition to the condition $0 \leq a \leq b \leq c \leq d$.

These results could be applied to many areas of mathematics. Especially in the complex analysis and algebra: In these areas of mathematics, inequalities of integrals are very important part. For example, we could be applied to these results for some localization of the zeros of some entire functions or some polynomials, like [1]- [4]. These results could be applied to many geometric surfaces like [5]-[7].

2. Related Results

Theorem1: *Let* $k = 1,2, \dots n$, $n \in \mathbb{N}$, $a_k, \varphi \in \mathbb{R}$, $a_k \in [0,1]$ $\varphi \in [0, \frac{\pi}{2}]$. *Then the function*

$$\left| \int_0^{e^{i\varphi}} x \prod_{k=1}^{n} (x + a_k) dx \right| \geq \frac{1}{n+2} \text{ for } n = 1,2,3.$$

Theorem2: *Let* $n \in \mathbb{N}$, $a \in \mathbb{R}, a \geq 0$. *The the function*

$$\left| \int_0^{e^{i\varphi}} (x + a)^n \, dx \right| \geq \frac{1}{n+1} \text{, whire } \varphi \in \left[0, \frac{\pi}{2}\right].$$

Theorem 3: *Let* $n \in \mathbb{N}$, $a \in \mathbb{R}$, $a \in [0,1]$. *Then the function*

$$\left|\int_0^i x(x+a)^n dx\right| \geq \frac{1}{n+2}.$$

We formulate Theorem1 for the case n=3 separately as Theorem4:

Therem4. *Let* $a, b, c, \varphi \epsilon \mathbb{R}$, $0 \leq a \leq b \leq c \leq 1, \varphi \in \left[0, \frac{\pi}{2}\right]$. *Then the function* $\left|\int_0^{e^{i\varphi}} x(x+a)(x+b)(x+c)dx\right| \geq 15$.

Proof: Let us put $B = \int_0^{e^{i\varphi}} x(x+a)(x+b)(x+c)dx$, $z = e^{i\varphi}$.

Then we obtain

$$5B = 5\int_0^z x(x+a)(x+b)(x+c)dx =$$

$$= 5\int_0^z x(x^2+\sigma_1 x+\sigma_2)(x+c)dx =$$

$$= 5\int_0^z [x^4+(\sigma_1+c)x^3+(\sigma_2+c\sigma_1)x^2+c\sigma_2 x]dx =$$

$$= \left(x^5+\frac{5}{4}(\sigma_1+c)x^4+\frac{5}{3}(\sigma_2+c\sigma_1)x^3+\frac{5}{2}c\sigma_2 x^2\right)\Big|_0^z =$$

$$= z^5+\frac{5}{4}(\sigma_1+c)z^4+\frac{5}{3}(\sigma_2+c\sigma_1)z^3+\frac{5}{2}c\sigma_2 z^2 = h(z).$$

Here we have put $\sigma_1 = a+b$, $\sigma_2 = ab$.

But according to the proof of Theorem3

$$4A = 4\int_0^z x(x+a)(x+b)dx =$$

$$= 4\int_0^z (x^3+\sigma_1 x^2+\sigma_2 x)dx =$$

$$= z^4+\frac{4}{3}\sigma_1 z^3+2\sigma_2 z^2 = f(z).$$

Let us divide $h(z)$ to $f(z)$:

$$z^5+\frac{5}{4}(\sigma_1+c)z^4+\frac{5}{3}(\sigma_2+c\sigma_1)z^3+\frac{5}{2}c\sigma_2 z^2 \quad \Big| \; z^4+\frac{4}{3}\sigma_1 z^3+2\sigma_2 z^2$$

$$- \quad z^5+\frac{4}{3}\sigma_1 z^4+2\sigma_2 z^3 \qquad\qquad z+\left(\frac{5}{4}c-\frac{\sigma_1}{12}\right)$$

$$\left(\frac{5}{4}c-\frac{\sigma_1}{12}\right)z^4+\left(\frac{5}{3}c\sigma_1-\frac{\sigma_2}{3}\right)z^3+\frac{5}{2}c\sigma_2 z^2$$

$$- \quad \left(\frac{5}{4}c-\frac{\sigma_1}{12}\right)z^4+\left(\frac{5}{3}c\sigma_1-\frac{\sigma_1^2}{9}\right)z^3+\left(\frac{5}{2}c\sigma_2-\frac{\sigma_1\sigma_2}{6}\right)z^2$$

$$\left(\frac{\sigma_1^2-3\sigma_2}{9}\right)z^3+\frac{\sigma_1\sigma_2}{6}z^2$$

Here we have put

$q(z) = z + \frac{5}{4}c - \frac{\sigma_1}{12}$,

$r(z) = \frac{\sigma_1^2 - 3\sigma_2}{9} z^3 + \frac{\sigma_1\sigma_2}{6} z^2$.

Then we assert $h(z) = f(z).q(z) + r(z)$.

According to Theorem3, we know $|A| \geq \frac{1}{4}$, i.e.$|f(z)| = 4|A| \geq 1$.

$5|B| = |h(z)| = |f(z)q(z) + r(z)| \geq |f(z)|.|q(z)| - |r(z)|$.

Then we need to prove that $|f(z)|.|q(z)| - |r(z)| \geq 1$. But we will prove that $|q(z)| \geq 1 + |r(z)|$, which is sufficiently.

We get:

$$|q(z)| = \left|z + \frac{5}{4}c - \frac{\sigma_1}{12}\right| \geq \left|e^{i\varphi} + \frac{5}{4}c - \frac{c}{6}\right| = \left|e^{i\varphi} + \frac{13}{12}c\right|,$$

because $0 \leq a \leq b \leq c$.

$$|r(z)| = \left|\frac{\sigma_1^2 - 3\sigma_2}{9} z^3 + \frac{\sigma_1\sigma_2}{6} z^2\right| \leq \left|\frac{(a+b)^2 - 3ab}{9}\right| + \left|\frac{(a+b).ab}{6}\right| \leq$$

$$\leq \left|\frac{a^2 + b^2 - ab}{9}\right| + \frac{b^3}{3} \leq \frac{c^2}{9} + \frac{c^3}{3},$$

because $0 \leq a \leq b \leq c$.

$$|q(z)| \geq 1 + |r(z)| \iff |q(z)|^2 \geq (1 + |r(z)|)^2 \geq \left(1 + \frac{c^2}{9} + \frac{c^3}{3}\right)^2 = \left(1 + \frac{4}{9}c^2\right)^2,$$

because $c \in [0,1]$

$$|q(z)|^2 \geq \left|e^{i\varphi} + \frac{13}{12}c\right|^2 \geq 1 + \left(\frac{13}{12}c\right)^2.$$

We will prove that $1 + \left(\frac{13}{12}c\right)^2 \geq \left(1 + \frac{4}{9}c^2\right)^2$. It`ll be true if $1 + \frac{169}{144}c^2 \geq 1 + \frac{8}{9}c^2 + \frac{16}{81}c^4$, i.e.$\left(\frac{169}{144} - \right.$ *89c2≥1681c4*, i.e. *41144c2≥1681c4*, *4116 c2=169c4*or *369≥256c2*. Therefore *369≥256c2*, i.e. *qz≥1+rz*, which confirms $5|B| \geq 1$, i.e.

$$\left|\int_0^{e^{i\varphi}} x(x+a)(x+b)(x+c)dx\right| \geq \frac{1}{5}.$$

3. Main Results

Theorem5. *Let* $a, b, c, d, \varphi \epsilon \mathbb{R}$, $0 \leq a \leq b \leq c \leq d \leq 1$, $\varphi \in [0, \frac{\pi}{2}]$. . *Then the function* $K(z) = \int_0^z x(x+a)(x+b)(x+c)(x+d)dx$, *where* $z = e^{i\varphi}$ *satisfies the inequality* $|K(z)| \geq \frac{1}{6}$

Proof:

I. Let $\varphi \in [0, \varphi_0]$, where $\cos\varphi_0 = 0{,}36$.

Let us calculate

$$6K(z) = 6\int_0^z x(x+a)(x+b)(x+c)(x+d)dx = 6\int_0^z (x^4 + \sigma_1 x^3 + \sigma_2 x^2 + \sigma_3 x)(x+d)dx =$$

$$= 6\int_0^z [x^5 + (\sigma_1 + d)x^4 + (\sigma_2 + d\sigma_1)x^3 + (\sigma_3 + d\sigma_2)x^2 + d\sigma_3 x]dx =$$

$$= \left[x^6 + \frac{6}{5}(\sigma_1 + d)x^5 + \frac{6}{4}(\sigma_2 + d\sigma_1)x^4 + \frac{6}{3}(\sigma_3 + d\sigma_2)x^3 + \frac{6}{2}d\sigma_3 x^2\right]\Big|_0^z =$$

$$= z^6 + \frac{6}{5}(\sigma_1 + d)z^5 + \frac{3}{2}(\sigma_2 + d\sigma_1)z^4 + 2(\sigma_3 + d\sigma_2)z^3 + 3d\sigma_3 z^2.$$

Here we put $\sigma_1 = a+b+c$, $\sigma_2 = ab+bc+ca$, $\sigma_3 = abc$. But according to the proof of Theorem4

$$5B = 5\int_0^z x(x+a)(x+b)(x+c)dx = 5\int_0^z (x^4+\sigma_1 x^3+\sigma_2 x^2+\sigma_3 x)dx =$$

$$= z^5 + \frac{5}{4}\sigma_1 z^4 + \frac{5}{3}\sigma_2 z^3 + \frac{5}{2}\sigma_3 z^2 = h(z).$$

Let us divide the polynomial $\mathbf{6K(z)}$ to the polynomial $\mathbf{f(z)}$:

$$\begin{array}{l|l} & z^5+\frac{5}{4}\sigma_1 z^4+\frac{5}{3}\sigma_2 z^3+\frac{5}{2}\sigma_3 z^2 \\ \hline z^6+\frac{6}{5}(\sigma_1+d)z^5+\frac{3}{2}(\sigma_2+d\sigma_1)z^4+2(\sigma_3+d\sigma_2)z^3+3d\sigma_3 z^2 & z+\left(\frac{6}{5}d-\frac{\sigma_1}{20}\right) \\ -\quad z^6+\frac{5}{4}\sigma_1 z^5+\frac{5}{3}\sigma_2 z^4+\frac{5}{2}\sigma_3 z^3 & \\ \hline \left(\frac{6}{5}d-\frac{\sigma_1}{20}\right)z^5+\left(\frac{3}{2}d\sigma_1-\frac{\sigma_2}{6}\right)z^4+\left(2d\sigma_2-\frac{\sigma_3}{2}\right)z^3+3d\sigma_3 z^2 & \\ -\quad \left(\frac{6}{5}d-\frac{\sigma_1}{20}\right)z^5+\left(\frac{3}{2}d\sigma_1-\frac{\sigma_1^2}{10}\right)z^4+\left(2d\sigma_2-\frac{\sigma_1\sigma_2}{12}\right)z^3+\left(3d\sigma_3-\frac{\sigma_1\sigma_3}{8}\right)z^2 & \\ \hline \left(\frac{\sigma_1^2}{16}-\frac{\sigma_2}{6}\right)z^4+\left(\frac{\sigma_1\sigma_2}{12}-\frac{\sigma_3}{2}\right)z^3+\frac{\sigma_1\sigma_3}{8}z^2 & \end{array}$$

i.e. $\mathbf{6K(z) = h(z).q_1(z) + r_1(z)}$, where we have put

$q_1(z) = z + \frac{6}{5}d - \frac{\sigma_1}{20}$,

$\mathbf{r_1(z) = mz^4 + nz^3 + lz^2}$,

$\mathbf{m = \frac{\sigma_1^2}{16} - \frac{\sigma_2}{6}, n = \frac{\sigma_1\sigma_2}{12} - \frac{\sigma_3}{2}, l = \frac{\sigma_1\sigma_3}{8}}$.

We obtain

$|6K(z)| = |h(z).q_1(z) + r_1(z)| \geq |h(z)|.|q_1(z)| - |r_1(z)|$.

Then we need to prove that $|h(z)|.|q_1(z)| - |r_1(z)| \geq 1$. But according to Theorem4, we know $|h(z)| \geq 1$, and therefore we will prove that $|q_1(z)| \geq 1 + |r_1(z)|$, which is sufficiently.

We get:

$$|\mathbf{q_1(z)}| = \left|\mathbf{z} + \frac{6}{5}\mathbf{d} - \frac{\mathbf{a+b+c}}{\mathbf{20}}\right| \geq \left|\mathbf{e^{i\varphi}} + \frac{\mathbf{21}}{\mathbf{20}}\mathbf{d}\right| \geq = \sqrt{\left(\cos\varphi + \frac{21}{20}d\right)^2 + \frac{441}{400}d^2} = \sqrt{1 + \frac{21}{10}d\cos\varphi + \frac{441}{400}d^2},$$

because $\mathbf{0 \leq a \leq b \leq c \leq d}$.

Obviously

$$|\mathbf{m}| = \left|\frac{\sigma_1^2}{16} - \frac{\sigma_2}{6}\right| = \left|\frac{(a+b+c)^2}{16} - \frac{(ab+bc+ca)}{6}\right| \leq \frac{9c^2}{16} - \frac{3c^2}{6} = \frac{c^2}{16} \leq \frac{d^2}{16},$$

$$|\mathbf{n}| = \left|\frac{\sigma_1\sigma_2}{12} - \frac{\sigma_3}{2}\right| = \left|\frac{(a+b+c)(ab+bc+ca)}{12} - \frac{abc}{2}\right| \leq \frac{3c.3c^2}{12} - \frac{c^3}{2} = \frac{c^3}{4} \leq \frac{d^3}{4},$$

$$|\mathbf{l}| = \left|\frac{\sigma_1\sigma_3}{8}\right| = \left|\frac{(a+b+c)abc}{8}\right| \leq \frac{3c^4}{8} \leq \frac{3d^4}{8},$$

and therefore

$$|\mathbf{r_1(z)}| = |\mathbf{mz^4 + nz^3 + lz^2}| \leq |\mathbf{m}| + |\mathbf{n}| + |\mathbf{l}| \leq \frac{\mathbf{d^2}}{\mathbf{16}} + \frac{\mathbf{d^3}}{\mathbf{4}} + \frac{\mathbf{3d^4}}{\mathbf{8}} \leq \mathbf{d^2}\left(\frac{1}{16} + \frac{1}{4} + \frac{3}{8}\right) = \frac{\mathbf{11}}{\mathbf{16}}\mathbf{d^2}.$$

Here we use $\mathbf{d \leq 1}$.

If $\varphi = \varphi_0$, then $\cos\varphi_0 = 0{,}36$ and we need to prove that

$$\sqrt{1+2{,}1.0{,}36d+\frac{441}{400}d^2} \geq \left(1+\frac{11}{10}d^2\right), i.e.$$

$$1+0{,}756d+1{,}1025d^2 \geq 1+\frac{11}{8}d^2+\frac{121}{256}d^4.$$

But

$$1+0{,}756d+1{,}1025d^2 \geq 1+0{,}756d^2+1{,}1025d^2 =$$

$$= 1+1{,}8585d^2 \geq 1+1{,}85d^2 \geq 1+\left(\frac{11}{8}+\frac{121}{256}\right)d^2 \geq 1+\frac{11}{8}d^2+\frac{121}{256}d^4,$$

which confirms $|6K(z)| \geq 1$, i.e $|K(z)| \geq \frac{1}{6}$.

II. Let $\varphi \epsilon \left[\varphi_0, \frac{\pi}{2}\right]$, where $\cos\varphi_0 = 0{,}36$.

First, we will prove that $m \geq 0,\ n \geq 0,\ l \geq 0$. For l it is obvious.

$$m = \frac{\sigma_1^2}{16} - \frac{\sigma_2}{6} = \frac{3\sigma_1^2 - 8\sigma_2}{48} \geq \frac{3.3\sigma_2 - 8\sigma_2}{48} = \frac{\sigma_2}{48} \geq 0,$$

because $\sigma_1^2 \geq 3\sigma_2$.

$$n = \frac{\sigma_1\sigma_2 - 6\sigma_3}{12} \geq \frac{3\sqrt[3]{abc}.3\sqrt[3]{(abc)^2} - 6\sigma_3}{12} = \frac{\sigma_3}{4} \geq 0,$$

because $a+b+c \geq 3\sqrt[3]{abc},\ ab+bc+ca = 3\sqrt[3]{(abc)^2}$.

Then we will explore $|r_1(z)|$:

$$|r_1(z)| = |mz^4+nz^3+lz^2| = |z^2|.|mz^2+nz+l| = \left|me^{2i\varphi}+ne^{i\varphi}+l\right| =$$

$$= \sqrt{(m\cos 2\varphi+n\cos\varphi+l)^2+(m\sin 2\varphi+n\sin\varphi)^2}$$

Since $\varphi \in \left[\varphi_0, \frac{\pi}{2}\right],\ \cos\varphi < 0$.

Then or $|m\cos 2\varphi+n\cos\varphi+l| \leq n\cos\varphi+l$

or $|m\cos 2\varphi+n\cos\varphi+l| \leq m|\cos 2\varphi|$.

Let us consider the function

$$f(\varphi) = (m\cos 2\varphi+n\cos\varphi+l)^2+(m\sin 2\varphi+n\sin\varphi)^2 =$$

$$= m^2+n^2+l^2+2mn(\cos 2\varphi.\cos\varphi+\sin 2\varphi.\sin\varphi)+2ml\cos 2\varphi+2nl\cos\varphi =$$

$$= m^2+n^2+l^2+2mn\cos\varphi+2ml(2\cos^2\varphi-1)+2nl\cos\varphi, i.e.$$

$$f(\varphi) = 2\cos\varphi(mn+nl)+4ml\ \cos^2\varphi+m^2+n^2+l^2-2ml$$

Since $\varphi \in \left[\varphi_0, \frac{\pi}{2}\right],\ m \geq 0, n \geq 0,\ l \geq 0$ we assert that $f(\varphi) \leq f(\varphi_0)$.

We obtain

$$n\cos\varphi_0+l \leq \frac{d^3}{4}.0{,}36+\frac{3d^4}{8}.$$

a) Let

$$|m\cos 2\varphi_0+n\cos\varphi_0+l| \leq n\cos\varphi_0+l.$$

Then

$$n\cos\varphi_0+l \leq \frac{0{,}36d^3}{4}+\frac{3d^4}{8} \leq \left(0{,}09+\frac{3}{8}\right)d^2 = 0{,}465d^2,$$

because $d \leq 1$.

$$m\sin 2\varphi_0+n\sin\varphi_0 \leq \frac{d^2}{16}.2\sin\varphi_0\cos\varphi_0+\frac{d^3}{4}\sin\ \varphi_0 \leq$$

$$\leq \frac{d^2}{8}.0{,}94.0{,}36+\frac{d^3}{4}.0{,}94 \leq 0{,}0423d^2+0{,}235d^3 \leq 0{,}2773d^2.$$

Then we obtain

$$f(\varphi) = (m\cos 2\varphi+n\cos\varphi+l)^2+\ (m\sin 2\varphi+n\sin\varphi)^2 \leq$$

$$\leq (0{,}456d^2)^2+(0{,}2773d^2)^2 \leq 0{,}294d^2, i.e.$$

$|r_1(z)| = \sqrt{f(\varphi)} \le 0,55d^2$. But we know $|q_1(z)| \ge \sqrt{1 + 1,8585d^2}$. Therefore we obtain

$$|q_1(z)|^2 \ge 1 + 1,8585d^2 \ge 1 + 1,4025d^2 \ge 1 + 2.0,55d^2 + (0,55d^2)^2 \ge 1 + |r_1(z)|^2,$$

which confirms the assertion ;

b) Let

$$|m\cos 2\varphi_0 + n\cos\varphi_0 + l| \le m|\cos 2\varphi_0|$$

$$|m\cos 2\varphi_0| = |m\,(2\cos^2\varphi_0 - 1)| = |0,7408m| \le 0,7408\frac{d^2}{16} = 0,0463d^2.$$

Then we get

$$f(\varphi) = (m\,cos\,2\varphi + n\cos\varphi + l)^2 + (m\sin 2\varphi + n\sin\varphi)^2 \le$$
$$\le (0,463d^2)^2 + (0,2773d^2)^2 \le 0,08d^4, i.e.$$

$|r_1(z)| = \sqrt{f(\varphi)} \le 0,2828d^2$. But we know from 2) that $|q_1(z)| \ge \sqrt{1 + 1,8585d^2}$.

Then we have

$$|q_1(z)|^2 = 1 + 1,8585d^2 \ge 1 + 0,6456d^2 \ge (1 + 0,2828d^2)^2 \ge (1 + |r_1(z)|)^2,$$

which completes the proof.

References

T.Stoyanov, "A localization of the zeros of some holomorphic functions "CPIOGI, "Proceedings of AIP" Applications of Mathematics in Engineering and Economics '34—AMEE '08, edited by M. D.Todorov, 2008 American Institute of Physics 978-0-7354-0598-l/08/

G.Polya, G. Szego, Problems and theorems in analysis I (Springer, Berlin, 1972) P.Maleev and S.L.Troyansky, On cotypes of Banachlattices.Constructive Function Theory'81. Sofia,1983,p.429-441.

T.Stoyanov, "Some extensions of Rolle's and Gauss-Lucas Theorems", in: Second international workshop, transform methods and special functions (Varna, August 23-30.1996)

T.Stoyanov, "About the zeros of some entire functions and their derivatives", Journal of the Australian Mathematical Society 68 (2000), [165-169].

G. H. Georgiev ,,Constructions of Rational Surfaces in the Three-dimensional Sphere, AIPConference Proceedings, Applications of Mathematics inEngineering and Economics(AMEE'11), Editors George Venkov, RalitzaKovachevaand VeselaPasheva, American Institute of Physics, Melville, NY, vol. 1410(2011), 213-220.

G. H. Georgiev,Geometric Transformations for Modeling of Curves and Surfaces, Shaker Verlag, Aachen, 2012.

G. H. Georgiev, Shape Curvatures of Planar Rational Spirals, In: Curves and Surfaces, LNCS, Vol. 6920, Springer (2012), 270-279.

T.Soyanov, "Inequalities of the modulus of some complex integrals", " Theory and Practice of modern science"-VII international workshop(Moskwa 2012), V.1,[10-16].

T.Soyanov, "Some estimates of the modulus of some complex integrals using division of polynomials", IV international scientific conference «Tendencies and prospects of development of modern scientific knowledge». (Moskwa 2012),[11,15].

T.Soyanov, "Some estimates below the modulus of integrals in the complex plane ", V-th international scientific conference «Tendencies and prospects of development of modern scientific knowledge» (to appear).