Mathematical Modeling of Asian Carp Invasion in the Upper Mississippi River System

Hyunju Oh, Ph.D.
Department of Mathematics and Computer Science
Bennett College
900 East Washington Street, Greensboro, NC, 27401
USA

Abstract

In our research we constructed two mathematical models, a competition model and a Lolka-Volterra model, with Asian Carp and native fish in the Upper Mississippi River System and the Great Lakes. The Asian Carp now threatens the Upper Mississippi River and the Great Lakes since they were imported from China in the 1970s to improve water quality of aquaculture and to control aquatic vegetation ponds. The goal of our research was to gain a better understanding of the interaction between Asian Carp and native species using mathematical model. In order to construct our research we used data from the Upper Midwest Environmental Science Center and then used numerical method with Maple to solve the system of partial differential equations which cannot be solved exactly. From our research we were able to find equilibriums which determine the behavior of the model such as coexistence or competition exclusion.

Keywords: Competition model, Lolka-Volterra model, Asian Carp, Upper Mississippi River System, Equilibriums, Mathematical model; Invasion

1. Introduction

In 1970s Asian Carp including Bighead Carp and Silver Carp were imported from China to improve water quality of aquaculture ponds and to control aquatic vegetation in the Mississippi river along Illinois. But they eat up the algae and other microscopic organism which are the important source of food for small fish and big fish in their ponds. The Asian Carp can grow incredibly quickly up to 110 pounds and the growth rate of population increases exponentially (NPS [Online]). An Asian Carp could eat 5-20% of its body weight each day, so the native fish whose diet overlaps with the diet of Asian Carp compete directly in the Mississippi river system. It causes to decrease the population of native fish quickly in the upper Mississippi river system (UMRS)(USGS [Online]).

Thus, they are considered invasive species, highly detrimental to the ecological balance, as they threaten the native fish population in the upper Mississippi river system and the Great Lakes. In July 2012 Congress enacted the "Stop Invasive Species Act", which requires the U.S. Corp of Engineers to implement measures, which will prevent Asian Carp from invading the Great Lakes from the Mississippi through the Chicago area canal system (Obama Administration Releases 2013 Asian Carp Control Strategy Framework).

The objective of this project is to make a mathematical model of this ecology problem and to discuss the qualitative behavior of the model. First, we make a mathematical model of competition model with parameters of Asian Carp, Silver Carp, and native fish species, Gizzard Shad for the same resources in the Open River (Figure 1).The system of partial differential equations in competition model cannot be solved exactly, so we use numerical method to solve them and investigate an equilibrium analysis. The equilibriums determine the behavior of the model such as coexistence or competitive exclusion. Second, we make a Lotka-Volterra model as predator-prey using Largemouth Bass as predator and Gizzard Shad as prey which is a competing species with Asian Carp in the Open River. We have lots of valuable ecological information from the equilibrium analysis of the model. The data for the model were obtained from U.S. Geological Survey (USGC), Upper Midwest Environmental Science Center (USGS [Online]). The Long Term Resource Mentoring Program (LTRMP) has been collecting Asian Carp and native fishes from multiple aquatic habitats of six reaches of the UMRS (Figure1)since 1993 (Koel *et al.* 2000).

2. Competition Model: Two Species of Silver Carp and Gizzard Shad in Open River

2.1 Materials

We select two fish species of Silver Carp and Gizzard Shad in Open River (Figure1) to create a competition model. The Silver Carp and Gizzard Shad consume phytoplankton and zooplankton, so they are competitors (Sampson *et al* 2009). We collect data of total catches of Silver Carp and Gizzard Shad in Open River from LTRMP (USGS [Online] and Pool Open River [Online]). We make a percentage table of each fish species total catches from 1993 to 2010 in Open River (Table 1). We can see clearly that Silver Carp population increases exponentially, but Gizzard Shad decreases quickly (Figure 2 & Figure 3).

2.2 Method: Mathematical Model of Competition

For an individual species with a population at time t given by $P(t)$, we can write general model using a differential equation of the form$\frac{dP}{dt} = rP(1-\frac{P}{M})$ where $r$is proportionality constant and$M$is a positive constant. This is called the logistic growth model (Brannan *et al*. 2010).

We make logistic growth models for Silver Carp and Gizzard Shad by numerical method using MAPLE. The differential equation of Silver Carp is $\frac{dS(t)}{dt} = 2.810S(t)\left(1-\frac{S(t)}{3.333728793}\right)$and the model of percentage of Silver Carp's population is $S(t) = \frac{2.7}{80990391483+919009608517e^{-2.81t}}$where S(0) = 0.27 with t = 0 representing year 2000. We apply a nonlinear least squares best fit to the data. The method of finding the best model to fit the data is computed by minimizing the expression given by the formula

$$V(P_0, M, r) = \sum_{i=0}^{n}(P(i) - data[i+1])^2.$$

Finally, we have best fit model of Silver Carp

$$\frac{dS(t)}{dt} = 0.33S(t) - 0.01S(t)^2, \; S(0) = 0.27. \quad \text{(eq.1)}$$

Similarly, we build a differential equation of Gizzard Shad,

$$\frac{dG(t)}{dt} = 0.007895G(t)(1-\frac{G(t)}{326.8885631.333728793})$$

And the model of percentage of Gizzard Shad's population $G(t) = \frac{42255}{1.57\times10^{12}-3.21\times10^{12}e^{-0.07894809277t}}$ where $G(0) = 33.804$ with t = 0 representing 1993 and the best model to fit the data by minimizing least squares method

$$\frac{dG(t)}{dt} = 25G(t) - G(t)^2, G(0) = 33.804. \quad \text{(eq. 2)}$$

Finally, we create a mathematical model for two competing species. We add the interspecies competition term, $-a_3S(t)G(t)$, to the (eq. 1), so the equation for the dynamics of the Silver Carp is given by $\frac{dS(t)}{dt} = 0.33S(t) - 0.01S(t)^2 - a_3S(t)G(t)$.

Similarly, the growth of Gizzard Shad satisfies the differential equation
$\frac{dG(t)}{dt} = 25G(t) - G(t)^2 - b_3S(t)G(t)$.
The system of differential equations cannot be solved exactly, so we use numerical method to find the unknown parameter using MAPLE. The computation of the sum of squares error is given by the formula

$$\sum_{i=0}^{n}((S(i) - dataS[i+1])^2 + (G(i) - dataG[i+1])^2).$$

The result of this complex minimization problem produces the parameters $a_3 = 1.4, b_3 = 9$ where$S(0) = 0.27$ $and G(0) = 27.892$ .Finally, we have the system of differential equations with the best fitting initial conditions is given by

$$\frac{dS(t)}{dt} = 0.33S(t) - 0.01S(t)^2 - 1.4S(t)G(t), \frac{dG(t)}{dt} = 25G(t) - G(t)^2 - 9S(t)G(t),$$

where $S(0) = 0.27, G(0) = 27.892$.

The equilibria are found by setting the derivatives equal to zero. To solve the systems of nonlinear equations, we use MAPLE and find the following the four equilibria:

$$[S = 0, G = 0], [S = 33, G = 0], [S = 0, G = 25], [S = 2.753772836, G = 0.2160444797].$$

In Figure 4, we can see two phase trajectories of the differential equations on the direction field. From the direction field, the mixed equilibrium solution $[S, G] \approx [2.753772836, 0.2160444797]$ is a saddle point, so unstable. Thus one species (Silver Carp) will eventually overwhelm the other (Gizzard Shad) and drive it extinction.

2.3 Results and Discussion

The surviving species is determined by the initial state of the system. A graph of data and the solutions to these differential equations is shown below (Figure 5).With initial conditions S(0)=0.27 and G(0)=27.9 in Figure 4, Silver Carp will be extinct around after one year and the Gizzard Shad is decreasing quickly and then stable to 25%.We investigate the model with several different initial conditions, and we can estimate of the value for this coexistence equilibrium. The Silver Carp will be extinct around after two years and the Gizzard Shad is increasing exponentially and then stable to 25%with initial conditions S(0)=2.5 and G(0)=0.1 (Figure 6). The Gizzard Shad will be extinct around after two and half years and the Silver Carp is increasing exponentially with initial conditions S(0) =2.6 and G(0) =0.1 (Figure 7). We can find that the Silver Carp will eventually overwhelm the Gizzard Shad when the percentage of the population of Silver Carp is greater than or equal to 2.6% and the percentage of the population of the Gizzard Shad is less than or equal to 0.1%.

3. The Lolka-Volterra Model: Largemouth Bass and Gizzard Shad in Open River

3.1. Materials

We select two fish species of Largemouth Bass and Gizzard shad in Open River (Figure1) to create a Lolka-Volterra model. We set Largemouth Bass species as a predator and Gizzard Shad species as a prey since Largemouth Bass primarily eat sunfish, shad and crayfish (Sampson *et al* 2009). We collect data of total catches of Largemouth Bass and Gizzard Shad in Open River from LTRMP (USGS [Online] &Pool Open River [Online]). We make a percentage table of each fish species total catches from 1993 to 2010 in Open River (Table 2). We can see clearly that the Gizzard Shad (prey) population begins dropping while the Largemouth Bass (predator) population is sufficiently high from the Figure 8 and 9. After the Gizzard Shad population falls, and then the predator Largemouth Bass population falls, this allows the prey Gizzard Shad population to recover and completes one cycle of this interaction. Thus, we see that qualitatively oscillations occur. Thus we can make the Lotka-Volterra, predator-prey, model for the dynamics of the populations of Largemouth Bass and its prey species Gizzard Shad.

3.2 Method: Mathematical Model of Lotka-Volterra

Let G(t) be the percentage of population of Gizzard Shad and L(t) be the percentage of population of Largemouth Bass. We develop a mathematical model based on the growth rates for the percentage of populations. The rate of change of the percentage of Gizzard Shad population is $\frac{dG(t)}{dt}$ and the rate of change of the percentage of Largemouth Bass population is $\frac{dL(t)}{dt}$. The percentage of Gizzard Shad population grows in proportion to its own population, denoted by $a_1 G$. The primary loss of Gizzard Shad is due to predation of Largemouth Bass. Predation is modeled by assuming random contact between the species in proportion to their populations with a fixed percentage of those contacts resulting in death of the prey species, denoted by $-a_2 G(t)L(t)$. The primary growth for the Largemouth Bass population depends on sufficient food for rising young Largemouth Bass, which implies an adequate source from predation on Gizzard Shad. Thus, the growth of percentage of the Largemouth Bass population is similar to the death rate for the Gizzard Shad population with different constant of proportionality, denoted by $b_2 G(t)L$. The loss of Largemouth Bass is in proportion to their own population, denoted by $-b_1 L(t)$.

The discussion leads to the Lotka-Volterra model:

$$\frac{dG(t)}{dt} = a_1 G(t) - a_2 G(t)L(t), \frac{dL(t)}{dt} = -b_1 L(t) + b_2 G(t)L(t),$$

where a_1, a_2, b_1, b_2 are positive constants.

The model ignores the role of climate variation and the interactions of other species, including human disturbance. We find the least squares best fit to the data (Table 2). We set the initial conditions G(0) = 33.8 and L(0) = 0.2. To find the rate constants a_1, a_2, b_1 and b_2, we use the averaging data of one period. Thus we obtain a reasonable estimate of equilibria for these data. The averaging the data between 2002 and 2008(omitting the first year to not bias the maximum) for the Gizzard Shad is 20.52 and the equilibrium estimates $G_e = \frac{b1}{b2} = 20.52$. Similarly, averaging the data between 1999 and 2010(omitting the last year to not bias the maximum) for the Largemouth Bass is 0.15 and the equilibrium estimates $H_e = \frac{a1}{a2} = 0.15$. From the graph of the data, we find a low in Largemouth Bass around 2007 and the Gizzard Shad population grows at this time (Table2). We get the $a_1 = 0.9423093823$ from $G(t) = G_0 e^{a_1 t}$, where $G_0 = 9.742$ in 2007, $G = 24.997$ in 2008. Since the steepest decline of the Largemouth Bass population occurs between 2003 and 2004 (Table 2), we get $b_1 = 1.791759469$ from $L(t) = L_0 e^{-b_1 t}$ where $L_0 = 0.12$ in 2003, $L = 0.02$ in 2014. Combine the above information; we obtain the initial estimates for the parameters

$a_1 = 0.9423093823, a_2 = 6.282062549, b_1 = 1.791759469$, and $b_2 = 0.08731771291$.

With these estimates on the parameters, we use MAPLE to find the least squares best fit of the model to the data. The formula is given by

$$V(a1, b1, a2, b2) = \sum_{i=0}^{n} \{(G(i) - dataG[i+1])^2 + (L(i) - dataL[i+1])^2\}.$$

The output of this function gives the best parameter value as $G(0) = 22$ and $L(0) = 0.9$. And $a_1 = 2.1, a_2 = 6, b_1 = 20.5$ and $b_2 = 1.1$. Finally, we have Lotka-Volterra model for Gizzard Shad and Largemouth Bass:

$$\frac{dG(t)}{dt} = 2.1G(t) - 6G(t)L(t), \frac{dL(t)}{dt} = -20.5L(t) + 1.1G(t)L(t),$$

where $G(0) = 22$ and $L(0) = 0.9$.

We obtain the graph of the percentage of populations of Largemouth Bass and Gizzard Shad as functions of the year and its phase portrait (Figure10& 11).

3.3 Results and Discussion

If there are no Largemouth Bass and Gizzard Shad, the populations are certainly not going to increase. If Gizzard Shad and Largemouth Bass populations are approximate to18.6% and 0.35%, respectively, then Gizzard Shad are just enough to support a constant Largemouth Bass population of 0.35% in Figure 11. There are neither too many Largemouth Bass (which would result in fewer Gizzard Shad) nor too few Largemouth Bass (which would result in more Gizzard Shad).

4. Table and Figures

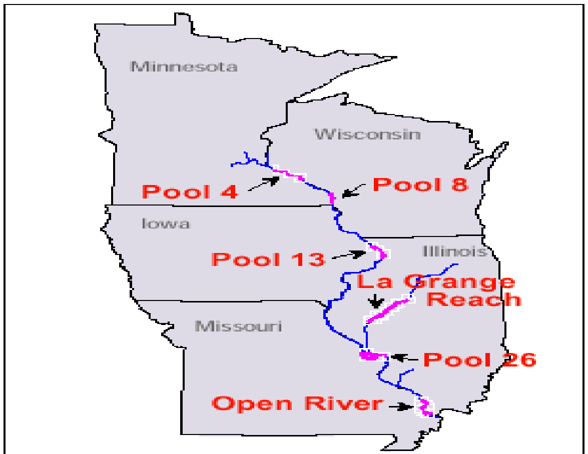


Figure 1. Upper Mississippi River System and locations of Long Term Resource Monitoring Program trend analysis areas.

Year	Total Catches of Silver Carp (Percentage)	Total Catches of Gizzard Shad/ Percentage	Total Catches by gear
1993	0 (0 %)	6057 (33.80 %)	17918
1994	0 (0 %)	1633 (11.61 %)	14061
1995	0 (0 %)	10860 (44.81 %)	24234
1996	0 (0 %)	5013 (35.42 %)	14154
1997	0 (0 %)	2097 (9.35 %)	22430
1998	0 (0 %)	5557 (37.03 %)	15006
1999	0 (0 %)	6987 (38.07 %)	18353
2000	32 (0.27 %)	3294 (27.89 %)	11810
2001	7 (0.05 %)	3839 (29.46 %)	13031
2002	13 (0.11 %)	5731 (48.55 %)	11804
2003	5 (0.1 %)	1230 (24.83 %)	4954
2004	121 (2.23 %)	1581 (29.17 %)	5420
2005	16 (0.32 %)	790 (15.97 %)	4948
2006	12 (0.21 %)	1061 (18.42 %)	5760
2007	248 (3.53 %)	684 (9.74 %)	7021
2008	318 (3.83 %)	2074 (25.00 %)	8297
2009	114 (2.54 %)	339 (7.54 %)	8297
2010	257 (5.1 %)	672 (13.34 %)	5039

Table 1: Total catches of fish collected in Pool Open River from 1993 – 2010

Figure 2 : Percentage of Silver Carp Total Catch in Open River

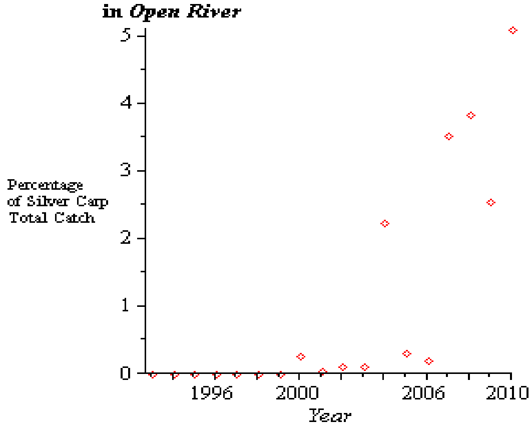


Figure 3: *Percentage of Gizzard Shad Total Catch in Open River*

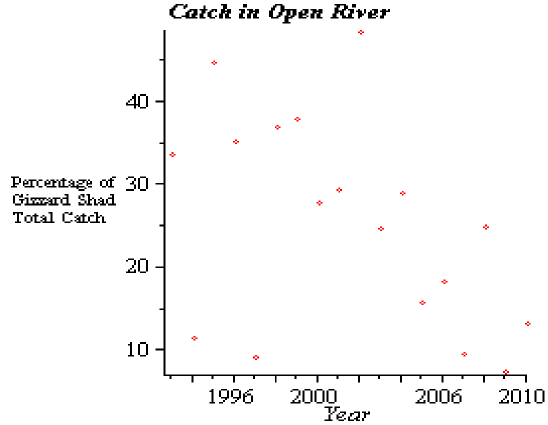


Figure 4: Phase Trajectories

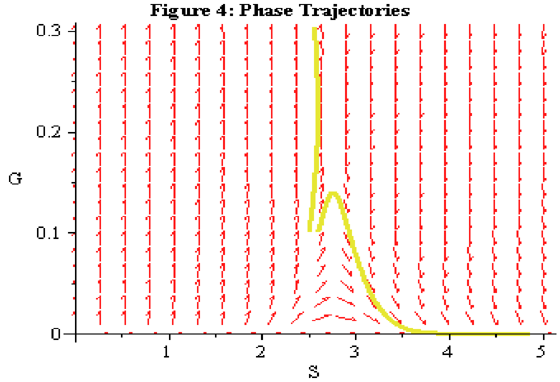
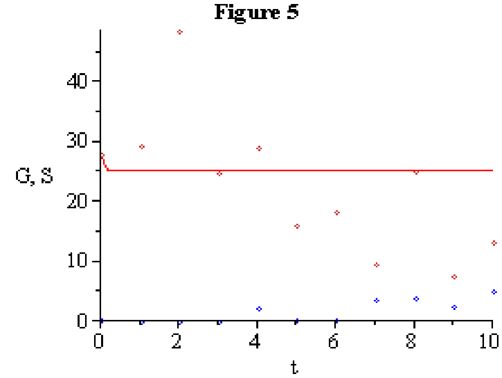
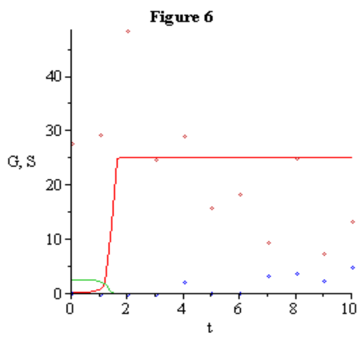
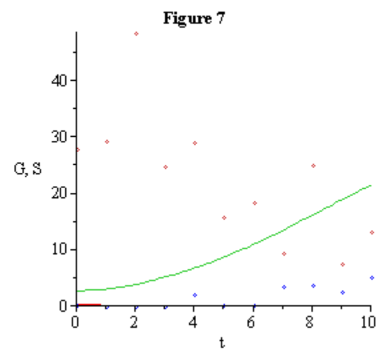


Figure 5

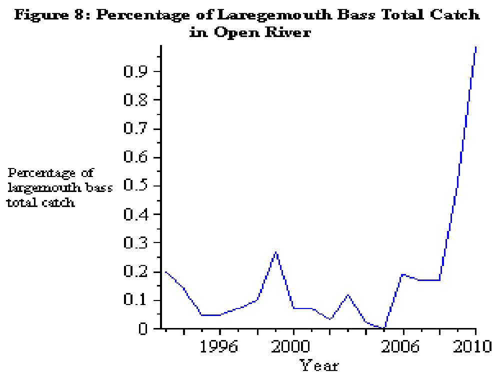


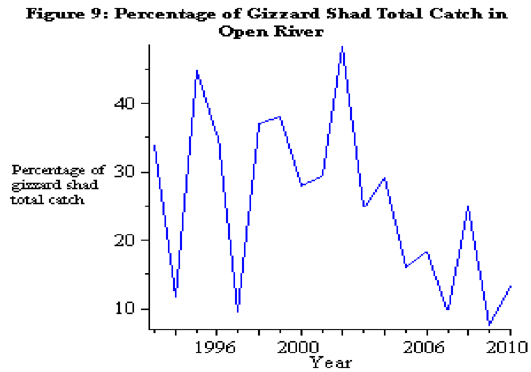




Year	Total Catches of Largemouth Bass (Percentage)	Total Catches of Gizzard Shad(Percentage)	Total Catches by gear
1993	36 (0.2 %)	6057 (33.80 %)	17918
1994	19 (0.14 %)	1633 (11.61 %)	14061
1995	11 (0.045 %)	10860 (44.81 %)	24234
1996	11 (0.045 %)	5013 (35.42 %)	14154
1997	16 (0.07 %)	2097 (9.35 %)	22430
1998	14 (0.1 %)	5557 (37.03 %)	15006
1999	49 (0.27 %)	6987 (38.07 %)	18353
2000	8 (0.07 %)	3294 (27.89 %)	11810
2001	9 (0.07 %)	3839 (29.46 %)	13031
2002	4 (0.034 %)	5731 (48.55 %)	11804
2003	6 (0.12 %)	1230 (24.83 %)	4954
2004	1 (0.02 %)	1581 (29.17 %)	5420
2005	0 (0 %)	790 (15.97 %)	4948
2006	11 (0.19 %)	1061 (18.42 %)	5760
2007	12 (0.17 %)	684 (9.74 %)	7021
2008	14 (0.17 %)	2074 (25.00 %)	8297
2009	23 (0.51 %)	339 (7.54 %)	8297
2010	50 (0.99 %)	672 (13.34 %)	5039

Table 2: Total catches of fish collected in Pool Open River from 1993 – 2010





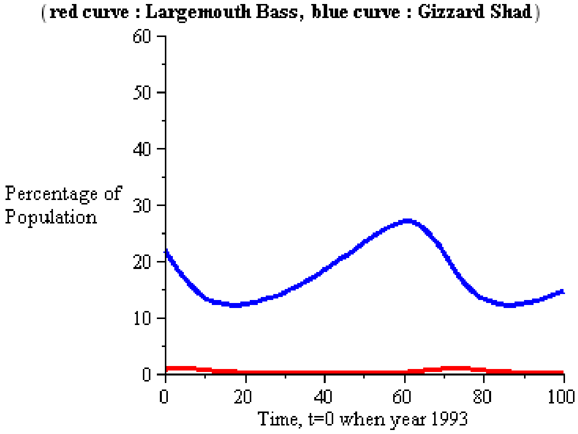


Figure 10 : Percentage of Population ofLargemouth Bass and Gizzard Shad
(**red curve : Largemouth Bass, blue curve : Gizzard Shad**)

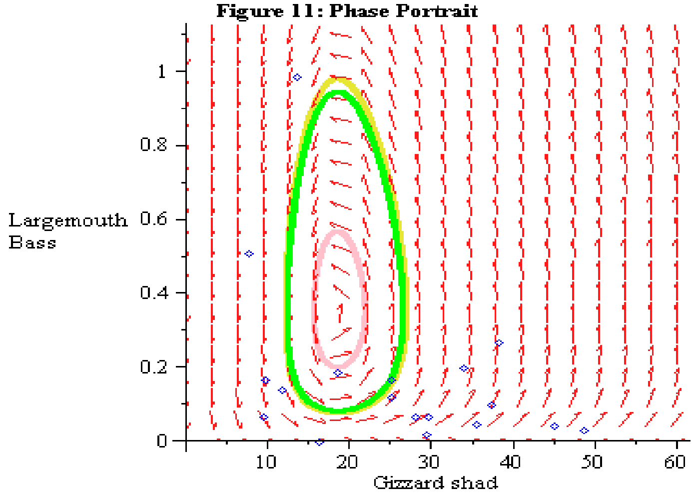


Figure 11: Phase Portrait

5. References

Brannan, J. & Boyce, W. (2010), Differential Equations with Boundary Value Problems, Wiley (Chapter 7)

Koel, T. M., Irons K.S., & Ratcliff, E. (2000). Asian Carp invasion of the upper Mississippi River System. Project Status Reports (PSR) 2000-05. [Online] Available: http://www.umesc.usgs.gov/reports_publications/psrs/psr_2000_05.html

National Park Service (NPS). [Online] Available: http://www. Nps.gov/miss/nature science/ascarpover.htm

Obama Administration Releases 2013 Asian Carp Control Strategy Framework. Council on Environmental Quality. [Online] Available: http://www.whitehouse.gov/administration/eop/ceq/Press_Releases/July_24_2013

Pool Open River (OR).[Online] Available: http://www.umesc.usgs.gov/data_library/fisheries/graphical/fish_total_catch.shtml

Sampson S. J., Chick J.H., &Pegg M.A. (2009).Diet overlap among two Asian carp and three native fishes in backwater lakes on the Illinois and Mississippi rivers. Biological Invasions, 11(3):483-496 U.S. Geological Survey (USGC).[Online]Available: http://www.umesc.usgs.gov/invasive_species/asian_carp.html