Semi-symmetric molecules and their symmetry operations with Clifford algebra

Sadiye CAKMAK
Abidin KILIC*
Physics Department, Anadolu University
Eskisehir, Turkey
E-mail: abkilic@anadolu.edu.tr*

Abstract

The Clifford algebra produces the new fields of view in the atom and mathematical physics, definition of bodies and rearranging for equations of mathematics and physics. The new mathematical approaches play an important role in the progress of physics. After presenting Clifford algebra and quaternions, the symmetry operations in molecular physics with Clifford algebra and quaternions are defined. This symmetry operations are applied to some symmetric and semi-symmetric solids too. Also, the vertices of some symmetric semisymmetric solids presented in the Cartesian coordinates are calculated.

1. Introduction

The ideas and concepts of physics are best expressed in the language of mathematics.But this language is far from unique. Many different algebraic systems exist and are in use today, all with their own advantages and disadvantages. In this paper we describe what we believe to be the most powerful available mathematical system developed to date. This is *geometric algebra*, which is presented as a new mathematical tool to add to your existing set as either a theoretician or experimentalist. Our aim is to introduce the new techniques via their applications, rather than as purely formal mathematics. These applications are diverse, and throughout we emphasis the unity of the mathematics underpinning each of these topics.

The history of geometric algebra is one of the more unusual tales in the development of mathematical physics. William Kingdom Clifford introduced his geometric algebra in the 1870s, building on the earlier work of Hamilton and Grassmann. It is clear from his writing that Clifford intended his algebra to describe the geometric properties of vectors, planes and higher-dimensional objects. But most physicists first encounter the algebra in the guise of the Pauli and Dirac matrix algebras of quantum theory. Few then contemplate using this unwieldy matrices for practical geometric computing. Indeed, some physicists come away from a study of Dirac theory with the view that Clifford's algebra is inherently quantum-mechanical [1].

Wessel, Argand and Gauss used the complex numbers in the solutions of two-dimensional problems. The exponential form of complex numbers is useful in the theory of rotational motions. The quaternion algebra, which was defined by Sir W. R. Hamilton, was generalized for the three dimensional complex numbers [2]. The quaternion algebra is the Clifford algebra of the two-dimensional anti-Eucliedean space. Quaternions in the three-dimensional spaces have more useful appearances for the subalgebras of Clifford algebra.In the n-dimensional spaces, Grassmann carried on the studies for the multi-dimensional bodies and defined the central product, which includes the both interior and exterior products. The Grassmann's central product is the Clifford product of vectors. Clifford tried to combine the Grassmann's algebra and quaternions in a mathematical system. Then this study, which was entitled "Application of Grassmann's Extensive Algebra", was published[3].Today, Clifford algebra has an important role in the investigations of the symmetry properties of systems, crystallography, molecular and solid state physics.

The method of point groups in the multi-dimensional spaces is derived by transforming in to the parameters of the reflections and possible rotational operations. In the three-dimensional spaces, Altmann showed that the Euler's angles are not useful for the rotational operations but the Euler-Rodriques' parameters are more advantageous [4]. To know the rotational pole and angle for each rotational operations is necessary and enough. So, there is a similarity between the proper and improper operations in P^3. The rest of paper is organized within 4 sections. Section 2 reveals the quaternions and symmetry operations in P^3. In section 3, the Clifford algebra and symmetry operations are showed. An application of the symmetry operations with Clifford algebra is given in section 4. Conclusions are drawn in the last section [5].

2. Quaternions and Symmetry Operations in R^3

The abstract quaternion group, discovered by William Rowan Hamiton in 1843, is an illustration of group structure [6].A quaternion is a quantity represented symbolically by **A** and it is defined through the following equations

$$\mathbf{A} = a1 + A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}, \quad (1)$$

or

$$\mathbf{A}=[a,\vec{\boldsymbol{A}}],\ \vec{A}= (\mathbf{A_x}, \mathbf{A_y}, \mathbf{A_z}), \quad (2)$$

where all a, A_x, A_y, $A_z$coefficients are the real numbers. The unitary quaternions **i, j, k**satisfy the multiplication rules as follows:

$$\mathbf{i\,j = k,\ \ j\,i= -k,\ jk=i,\ \ k\,j= -i,\ \ i\,k= -j,\ \ k\,i=j.} \quad (3)$$

Also **i, j** and **k** can be written as $\mathbf{e_1},\mathbf{e_2}$and $\mathbf{e_3}$, reciprocally. The vector quaternion**A** with components [0, A_x, A_y, A_z] and a vector $\vec{A}$ of the Euclidean tridimensional space with components (A_x, A_y, A_z) are reciprocally associated [7].

If A and B quaternions are

$$A =a1+A_xe_1 + A_ye_2 + A_ze_3 = [a, \vec{\boldsymbol{A}}], \quad (4)$$

and

$$B =b1+B_{xe1} + B_ye_2 + B_ze_3 = [b, \vec{B}\,], \quad (5)$$

the product of two quaternions, namely A and B, is given by

$$AB =[ab -\vec{A}\,.\,\vec{B}\,,\ a\vec{B} +b\vec{A} +\vec{A}\times\vec{B}\,], \quad (6)$$

$$AB=(ab-A_xB_x - A_yB_y - A_zB_z) + e_1(A_xb + aB_x + A_yB_z - A_zB_y) + \quad (7)$$

$$e_2(aB_y + A_yb-A_XB_Z+A_ZB_X) + e_3(aB_z + A_zb+A_xB_y - A_yB_x), \quad (8)$$

where the result is a quaternion. It must be noted that the product of quaternions is not commutative, but associative. The product of **A** and **B** quaternions in the matrix form can be written as

$$AB=(1,e_1,e_2,e_3)\begin{bmatrix} a & -A_x & -A_y & -A_z \\ A_x & a & -A_z & A_y \\ A_y & A_z & a & -A_x \\ A_z & -A_y & A_x & a \end{bmatrix}\begin{bmatrix} b \\ B_X \\ B_Y \\ B_Z \end{bmatrix} \quad (9)$$

$$=(1,e_1,e_2,e_3)\begin{bmatrix} c \\ C_X \\ C_Y \\ C_Z \end{bmatrix} \quad (10)$$

We can calculate the quaternion product of A, B and C quaternions,

$$ABC=(1,e_1,e_2,e_3)\begin{bmatrix} a & -A_x & -A_y & -A_z \\ A_x & a & -A_z & A_y \\ A_y & A_z & a & -A_x \\ A_z & -A_y & A_x & a \end{bmatrix}\begin{bmatrix} b & -B_x & -B_Y & -B_Z \\ B_X & b & -B_Z & B_Y \\ B_Y & B_z & b & -B_x \\ B_z & -B_Y & B_X & b \end{bmatrix}\begin{bmatrix} c \\ C_X \\ C_Y \\ C_Z \end{bmatrix} \quad (11)$$

$$ABC=(1,e_1,e_2,e_3)\begin{bmatrix} 1 \\ D_x \\ D_y \\ D_z \end{bmatrix} \quad (12)$$

where**D** is equivalent to a quaternion. Also, each quaternion matrix is determined by the first column (or row) alone, which simplifies the construction of the corresponding (4x4)-matrix algebra considerably. For each quaternion A, its conjugate is [8]

$$A^*=al-A_xe_1-A_ye_2-A_ze_3 \quad (13)$$

The rotation of arbitrary points on an unit sphere ($\mathbf{R}(\gamma\mathbf{k})$) can be defined by γ angle around of the definite k-axis (0< γ <Π). The poles of rotational operations are defined on the every half-sphere. In general, the rotations (counterclokwise) are accepted to be positive direction. If $\mathbf{R}(\beta\mathbf{l})$ and $\mathbf{R}(\alpha\mathbf{m})$ are the rotations with *8* and *α* angles around of l and m axes, respectively, then the result of two rotational operations has to be equal to a new rotation with γ-angle around of k- axis, *i.e.*

R(β**l**) **R**(α**m**) = **R**(γ**k**), (14)

where k, l, m are the axial vectors that correspond to the k, l, m rotation axes, respectively. Rodriques [9] defined the geometrical structure and algebra for the angles γ, *β, α* and the axes k, l, m in the following forms

$\cos\frac{\gamma}{2} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{m}.\mathrm{l},$ (15)

$\sin\frac{\gamma}{2}k = \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\mathrm{m} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\mathrm{l} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\mathrm{m} \times \mathrm{l}$ (16)

The rotational operations with the normalized quaternions can be written as

R(α**m**)=$[\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}\mathbf{m}]$ (17)

R(β**l**)=$[\cos\frac{\beta}{2}, \sin\frac{\beta}{2}\mathbf{l}]$ (18)

In the similar way, from eqs. (14, 17, 18), **R**(γ**k**) can be defined by the following equations

R(γ**k**) = $[\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}\mathbf{m}]\ [\cos\frac{\beta}{2}, \sin\frac{\beta}{2}\mathbf{l}]$ (19)

R(γ**k**)= $[\cos\frac{\gamma}{2}, \sin\frac{\gamma}{2}\mathbf{k}]$ (20)

Any point represented by **R** quaternion transforms to the a new quaternionic point at the end of a rotation defined by **A** unit quaternion. This new quaternion is

R'=**ARA*,** (21)

where A* is complex conjugate of **A** [5].

3. Symmetry Operations with Clifford Algebra

Clifford algebra in physics is used for the studies of symmetry. There are three basic units e_i*(i=1,* 2, 3) in Clifford algebra such that

$\mathbf{e}_i\mathbf{e}_j+\mathbf{e}_j\mathbf{e}_i=2\delta_{ij}$**,** (22)

which are equivalent to

$\mathbf{e}_i\mathbf{e}_i=1$, (23)

$\mathbf{e}_i\mathbf{e}_j=-\mathbf{e}_j\mathbf{e}_i$. (24)

The easiest way to understand the geometric product is by example, so consider a two-dimensional space (a plane) spanned by two orthonormal vectors $\mathbf{e}_1$and $\mathbf{e}_2$**.** These basis vectors satisfy

$\mathbf{e}_1^2=\mathbf{e}_2^2=1$ (25)

$\mathbf{e}_1.\mathbf{e}_2=0$ (26)

The final entity present in the algebra is the bivector$\mathbf{e}_1\wedge\mathbf{e}_2$.This is the highest grade element in the algebra, since the outer product of a set of dependent vectors is always zero.

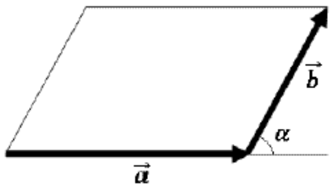

Figure 1. The geometrical meaning of **a**Λ**b** [11].

The Clifford product of two vectors, namely $\vec{a}$and $\vec{b}$, with components

$\vec{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$, (27)

and

$\vec{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$, (28)

is given by

$\vec{a}\vec{b} = \vec{a}.\vec{b} + \vec{a} \wedge \vec{b} = a_1b_1+a_2b_2+(a_1b_2-a_2b_1)\mathbf{e}_{12}$**,** (29)

where "Λ" is called as the wedge product [10]. To study the properties of the bivector$\mathbf{e}_1\Lambda\mathbf{e}_2$ we first recall that for orthogonal vectors the geometric product is a pure bivector:

$\mathbf{e}_1\mathbf{e}_2=\mathbf{e}_1.\mathbf{e}_2 +\mathbf{e}_1\Lambda\mathbf{e}_2=\mathbf{e}_1\Lambda\mathbf{e}_2=\mathbf{e}_{12}$ (30)

and that orthogonal vectors anticommute:

$\mathbf{e}_2\mathbf{e}_1=\mathbf{e}_2\Lambda\mathbf{e}_1=-\mathbf{e}_1\Lambda\mathbf{e}_2=-\mathbf{e}_1\mathbf{e}_2$ (31)

We can now form products in which $\mathbf{e}_2\mathbf{e}_1$ multiplies vectors from the left and the right. First from the left we find that

$(\mathbf{e}_1\Lambda\mathbf{e}_2)\mathbf{e}_1=(-\mathbf{e}_2\mathbf{e}_1)\mathbf{e}_1=-\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1=-\mathbf{e}_2$ (32)

and

$(\mathbf{e}_1\Lambda\mathbf{e}_2)\mathbf{e}_2=(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_2=\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2=\mathbf{e}_1$ (33)

If we assume that e_1 and e_2 form a right-handed pair, we see that left multiplication by the bivector rotates vectors 90° clockwise(i.e. in a negative sense).

Similarly, acting from the right

$\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2)=\mathbf{e}_2$, (34)

$\mathbf{e}_2(\mathbf{e}_1\mathbf{e}_2)=-\mathbf{e}_1$, (35)

So right multiplication rotates 90° anticlockwise a positive sense.The final product in the algebra to consider is the square of the bivector$\mathbf{e}_1\Lambda\mathbf{e}_2$:

$(\mathbf{e}_1\Lambda\mathbf{e}_2)^2=\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2=-\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2=-1$. (36)

Geometric consideration have led naturally to quantity which square to -1. This fits with the fact that two successive left (or right) multiplication of a vector by e_1e_2 rotates the vector through 180°, which is equivalent to multiplying by -1. 1, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_{12}$ form the basis of the Clifford algebra $Cl_2$of the vector plane R^2. The Clifford algebra $Cl_2$is the four-dimensional linear space and its basis elements have the multiplication table as follows:

Table 1. The basis of Clifford Algebra

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

The reflection of $\vec{r}$ across the line $\vec{a}$, namely the mirror image $\vec{r}$ of $\vec{r}$ with respect to $\vec{a}$ is given by

$\vec{r}=\vec{a}\ \vec{r}\ \vec{a}^{-1}$ (37)

Equation (37) can be directly obtained from using the commutation properties of Clifford algebra [10]. The composition of two reflections, first across $\vec{a}$ and then across $\vec{b}$, is given by

$\vec{r}=\vec{b}\ \vec{r}\ \vec{b}^{-1}=\vec{b}(\vec{a}\ \vec{r}\ \vec{a}^{-1})\ \vec{b}^{-1}=(\vec{b}\ \vec{a})\,\vec{r}\,(\vec{b}\ \vec{a})^{-1}$ (38)

The composite of these two reflections is a rotation by twice the angle between $\vec{a}$ and $\vec{b}$.

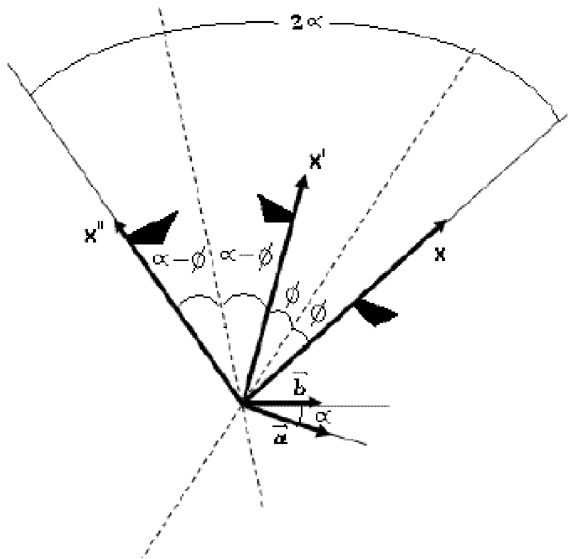

Figure 2. When **x** is subjected two successive reflection first with respect to a plane perpendicular to a and then with to a plane perpendicular to b, the result is a rotation of **x** about an axis in the direction axb. The angle of rotation is twice the angle between a and b [12].

Now, the symmetry operations in the three-dimensional space must be mapped on the elements of the Clifford algebra, Cl_3 [13]. The Clifford algebra Cl_3 of R^3 is the real associative algebra generated by the set of e_1, e_2,e_3, which are the eight-dimensional with the following basis

1	the scalar	
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	vectors	
$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$	bivectors	(39)
$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	a volume element	

An arbitrary element in Cl_3 is a sum of a scalar, a vector, a bivector and a volume element, and can be written as

$$u = u_0+u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 + u_{12}\mathbf{e}_{12} +u_{13}\mathbf{e}_{13} + u_{23}\mathbf{e}_{23} + u_{123}\mathbf{e}_{123}. \qquad (40)$$

The Clifford units, ei's, are identified with orthogonal reflections (mirrors)

$$\mathbf{e}_1\leftrightarrow\sigma_{yz},\ \mathbf{e}_2\leftrightarrow\sigma_{xz},\qquad \mathbf{e}_3\leftrightarrow\sigma_{xy}. \qquad (41)$$

The mappings between the Clifford bivectors $\mathbf{e}_i\mathbf{e}_j$ and the corresponding quaternion units are defined

$$\mathbf{e}_3\mathbf{e}_2 \leftrightarrow\sigma_{xy}\sigma_{xz} = C_{2x}\leftrightarrow [0, (1, 0, 0)], \qquad (42)$$

$$\mathbf{e}_1\mathbf{e}_3\leftrightarrow\sigma_{yz}\sigma_{xy} = C_{2y}\leftrightarrow [0, (0,1, 0)], \qquad (43)$$

$$\mathbf{e}_2\mathbf{e}_1 \leftrightarrow\sigma_{xz}\sigma_{yz} = C_{2z}\leftrightarrow [0, (0, 0,1)]. \qquad (44)$$

The inversion, which is a product of three reflections, is obtained by a trivector as follows:

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\leftrightarrow\sigma_{yz}\sigma_{xz}\sigma_{xy} = i, \qquad (45)$$

where $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \in Cl_3$ is performed [5].

4. An Application of The Symmetry Operations with Clifford Algebra

The reflection and rotation operations in the solid state physics and molecular physics play an important role. The symmetry operations can be easily applied on the regular polyhedra, which are called *Platonic Solids.*, and semi-regular polyhedra, which are called *Archimedean Solids.* The Platonic solids are tetrahedron, cube, octahedron, icosahedron and dodecahedron. Some important numbers for the Platonic solids are shown in Table 2. The Archimedean solids are truncated tetrahedron, cuboctahedron, truncated cube, truncated octahedron, rhombicuboctahedron (or small rhombicuboctahedron), truncated cuboctahedron (or great rhombicuboctahedron), snub cube (or snub hexahedron), icosidodecahedron, truncated dodecahedron, truncated icosahedron, rhombicosidodecahedron (or small rhombicosidodecahedron), truncated icosidodecahedron (or great rhombicosidodecahedron), snub dodecahedron (or snub icosidodecahedron).Some important numbers for the Platonic solids are shown in Table 3 [14].

Table 2. Numbers for the five Platonic solids

	Number of faces	*Number of edges*	*Number of vertices*	*Edges per face*
Tetrahedron	4	6	4	3
Cube	6	12	8	4
Octahedron	8	12	6	3
Icosahedron	20	30	12	3
Dodecahedro	12	30	20	5

A rhombicuboctahedron has twenty four-vertices. The vertices of a dodecahedron, whose origin was chosen at the centre of body, are indexed as cartesian coordinates in Table 3.

Table 3. Numbers for the thirteen Archimedean solids

	Number of faces	*Numbe r of*	*Number of vertices*
Truncated Tetrahedron	8(4 triangles,4 hexagons)	18	12
Cuboctahedron	14 (8 triangles,6 squares)	24	12
Truncated Cube	14 (8 triangles,6 octagons)	36	24
Truncated Octahedron	14 6 squares,8 hexagons)	36	24
Rhombicuboctahedron	26 (8 triangles,18 squares)	48	24
Truncated	26 (12 squares,8 hexagons,6	72	48
Snub Cube	38 (32 triangles,6 squares)	60	24
Icosidodecahedron	32 (20 triangles,12 pentagons)	60	30
Truncated Dodecahedron	32 (20 triangles,12 decagons)	90	60
Truncated Icosahedron	32 (12 pentagons,20 hexagons)	90	60
Rhombicosidodecahedro	62 (20 triangles,30 squares,12	120	60
Truncated	62 (30 squares,20 hexagons,12	180	120
Snub Dodecahedron	92 (80 triangles,12 pentagons)	150	60

Assuming the distances of midpoints of all the edges to the origin of the rhombicuboctahedronare normalized. We suppose that edge lenght is 2a.

Table 4. For the model, shown in Fig. 1, the vertices of a rhombicuboctahedron

corner	x	y	z
1	a	-a	1,85
2	a	a	1,85
3	-a	a	1,85
4	-a	-a	1,85
5	a	-1,8	a
6	1,8	-a	a
7	a	-1,8	-a
8	1,8	-a	-a
9	-a	1,8	a
10	-1,8	a	a
11	-a	1,8	-a
12	-1,8	a	-a
13	1,8	a	a
14	a	1,8	a
15	1,8	a	-a
16	a	1,8	-a
17	-1,8	-a	a
18	-a	-1,8	a
19	-1,8	-a	-a
20	-a	-1,8	-a
21	a	-a	-1,8
22	a	a	-1,8
23	-a	a	-1,8
24	-a	-a	-1,8

Now, for example, from eqs.(12), (13) and (21), the C_2 rotation of 13th vertex around the z-axis is

$$R' = A_z R_{13} a_z^{*} \tag{46}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1,85 & -a & -a \\ 1,85 & 0 & -a & -a \\ a & a & 0 & -1,85 \\ a & a & 1,85 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} 0 \\ -1,85 \\ -a \\ a \end{bmatrix} \tag{48}$$

where$\mathbf{A}_z$, R_{13} and $a_z^*$are the (4x4)-matrix representations of rotation z-axis and 13th vertex, (4×1) column matrix representation of the (4×4) matrix of A_z^* respectively. This result of the matrix operation defines the 17th vertex of the rhombicuboctahedron.

This method is the conventional geometrical method of the rotation operation for rhombicuboctahedron. Using quaternions, the same calculation can be obtained as follows

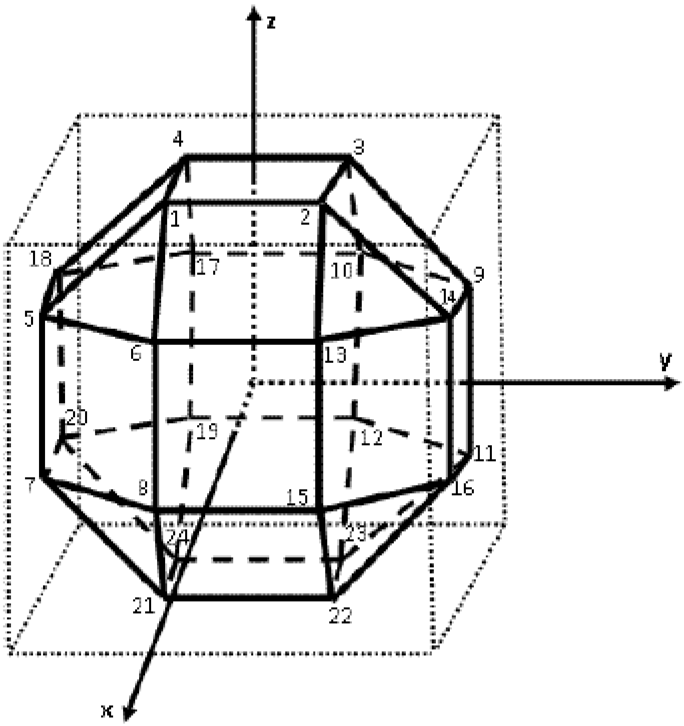

Figure 3.Rhombicuboctahedron.

$R'=a_z R_{13}a_z^*$ (49)
$= (1e_{12})(\mathbf{1{,}85e_1 + ae_2 + ae_3})(-le_{12})$ (50)
$= (-1{,}85e_1 + -ae_2 + ae_3).$ (51)

This result is equal to the quaternionic definition of 17th vertex, as well. Now we investigate another rotation operation of 17th z-axis around the straight line.The rotation 90° of 17th vertex z-axis around this straight line.

$R' = a_z R_{17}a_z^*$ (52)
$= (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}e_{12})(-1{,}85e_1 - ae_2 + ae_3)(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} e_{12})$ (53)
$= (-ae_1+1{,}85e_2+ae_3)$ (54)

This result is equal to the quaternionic definition of9th vertex. The rotation 45° of 9th vertex z-axis around this straight line

$R'=a_z R_9a_z^*$ (55)
$= (\frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2}e_{12})(-ae_1+ 1{,}85e_2 +ae_3) (\frac{\sqrt{\sqrt{2}+2}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} e_{12})$ (56)
$= (ae_1+ 1{,}85e_2+ae_3).$ (57)

This result is equal to the quaternionic definition of 14th vertex. The reflection operations can be also defined by the Clifford algebra elements. According to the xz-plane, the reflection of 2th vertex with Clifford algebra is

$R'=\sigma_{xz} R_{10}\sigma_{xy}^{-1}$ (58)
$=e_{13}(ae_1+ ae_2+ 1{,}85e_3)e_{13}$ (59)
$=ae_1- ae_2+ 1{,}85e_3,$ (60)

where□$^{-1}$ is the inverse of σ_{xz} operation. The reflection of 10th vertex on the xz-plane is equal to the 1th vertex. After this reflection of the 2th vertex on the xz-plane, the reflection of 1th vertex on the xy-plane can be written as

$R''=\sigma_{xy} R'\sigma_{xy}^{-1}$ (61)
$=\sigma_{xy} (\sigma_{xz}R_2\sigma_{xy}^{-1}) \sigma_{xy}^{-1}$ (62)
$= (\sigma_{xy}\sigma_{xz})R_2(\sigma_{xz}\sigma_{xy})$ (63)
$=ae_1-ae_2-1,85e_3$ (64)
This result is equal to the quaternionic definition of 21th vertex in Table 4.

The rotations 45° (C_8),90° (C_4), 180° (C_2) of around axis which are perpendicular to the this surface, from the midpoint of each square face and the rotations 60° (C_6)of around axis which are perpendicular to the this surface, from the midpoint of each face of triangle are possible.3 reflection in a plane perpendicular to a 3-fold axis and the rotoreflections 180° (S_2) of around the axis drawn from each corner to the other corner are also possible.O_h(*432) group; achiral octahedral symmetry or full octahedral symmetry; is made up of all this operations of rotation and reflection.

5. Conclusions

In this study we apply Clifford Algebra to Archimedean Solids.They are semi-symmetric solids. The geometrical methods and matrices are used for the investigation of symmetry operations of the symmetric solid too [5]. Clifford algebras are algebras of geometries and quaternions are hypercomplex numbers [15]. In this study, Clifford algebra and quaternions are used for the symmetry operations. When these operations are made with Clifford algebra and quaternions, it is obvious that the calculations are easy and compact. The quaternions and Clifford algebra are much simpler to apply to the symmetry operations than the conventional methods of molecular symmetry. This method can be applied to the more complex structures.

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