IMPULSIVE DIFFERENTIAL INEQUALITIES AND COMPARISON THEOREMS FOR POSITIVE SOLUTIONS OF NEUTRAL DELAY EQUATIONS

Isaac, I. O.
Zsolt Lipcsey

Department of Mathematics/Statistics and Computer Science
University of Calabar
P.M.B. 1115, Calabar, Cross River State
Nigeria.

Abstract

It has become imperative in recent times to determine the properties of the solutions of certain mathematical equations or inequalities from the knowledge of associated inequalities or equations. In this paper, a number of oscillatory properties of the solutions of impulsive differential inequalities are established using the knowledge of those of their associated equations and what is more, some comparison theorems for the positive solutions of certain neutral delay impulsive differential equations are also formulated.

KEY WORDS: Neutral delay impulsive differential inequalities and equations; Comparison results and theorems.

1. Introduction

Sometimes in analysis and in many other considerations, there exists the need to compare the properties of the solutions of certain mathematical equations and inequalities. Such comparison may take the form of conclusions about the behaviours of the solutions of the inequalities knowing the behaviours of the solutions of the associated equations and vice versa. In this paper, we

(i) establish some comparison results for positive solutions of a neutral delay impulsive differential equation of the form

$$\begin{cases} \left[x(t)+\sum_{j=1}^{M} p_j(t)x(t-\tau_j(t))\right]' + \sum_{i=1}^{N} q_i(t)x(t-\sigma_i(t)) = 0, t \neq t_k \\ \Delta\left[x(t_k)+\sum_{j=1}^{M} p_j(t_k)x(t_k-\tau_j(t_k))\right] + \sum_{i=1}^{N} q_{ik}x(t_k-\sigma_i(t_k)) = 0 \end{cases} \tag{1.1}$$

and the inequalities

$$\begin{cases} \left[y(t)+\sum_{j=1}^{M} a_j(t)y(t-\tau_j(t))\right]' + \sum_{i=1}^{N} b_i(t)y(t-\sigma_i(t)) \le 0, t \neq t_k \\ \Delta\left[y(t_k)+\sum_{j=1}^{M} a_j(t_k)y(t_k-\tau_j(t_k))\right] + \sum_{i=1}^{N} b_{ik}y(t_k-\sigma_i(t_k)) \le 0 \end{cases} \tag{1.2}$$

and

$$\begin{cases} \left[z(t)+\sum_{j=1}^{M} c_j(t)z(t-\tau_j(t))\right]' + \sum_{i=1}^{N} d_i(t)z(t-\sigma_i(t_k)) \ge 0, t \neq t_k \\ \Delta\left[z(t_k)+\sum_{j=1}^{M} c_j(t_k)z(t_k-\tau_j(t_k))\right] + \sum_{i=1}^{N} d_{ik}z(t_k-\sigma_i(t_k)) \ge 0, \end{cases} \tag{1.3}$$

where p_j, a_j, c_j, q_i, b_i and d_i are temporarily assumed to be piece-wise continuous functions; τ_j and σ_i are continuous and q_{ik}, b_{ik} and d_{ik} are positive constants for all $t,\ t_k \in R_+$, $1 \le j \le M$ and $1 \le i \le N$; and

(ii) formulate comparison theorems for neutral delay impulsive differential equations with the aid of some known oscillatory properties of the solutions of other given delay impulsive equations.

Unlike ordinary differential equations, the solution $x(t)$ for $t \in [t_0, T)$ of an impulsive differential equation or its first derivative $x'(t)$ is a piece-wise continuous function with points of discontinuity $t_k \in [t_0, T)$, $t_k \neq t$.

Therefore, in order to simplify the statements of our assertions later, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r \in N$, D := [T,∞) ⊂ R and let $S := \{t_k\}_{k\in E}$, where E represents a subscript set which can be the set of natural numbers N or the set of integers Z, be fixed. Throughout our discussion, we will assume that the sequences $\{t_k\}_{k\in E}$ are moments of impulse effect and satisfy the properties:

C1.1 If $\{t_k\}_{k\in E}$ is defined with $E := N$, then $0 < t_1 < t_2 < \cdots$ and

$$\lim_{k\to+\infty} t_k = +\infty;$$

C1.2 If $\{t_k\}_{k\in E}$ is defined with $E := Z$, then $t_0 \le 0 < t_1, \ \ t_k < t_{k+1}$ for all $k \in Z$, k≠0 and

$$\lim_{k\to\pm\infty} t_k = \pm\infty.$$

We denote by $PC(D, R)$ the set of all functions $\varphi : D \to R$, which are continuous for all $t \in D$, t∉S. They are continuous from the left and have discontinuity of the first kind at the points for which $t \in S$.

By $PC^r(D, R)$, we denote the set of functions $\varphi : D \to R$ having derivative $\frac{d^j\varphi}{dt^j} \in PC(D, R)$, $0 \le j \le r$ ({Bainov/Simeonov: 1998}; {Lakshmikantham et al: 1989}). To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols $PC(D, R; S)$ and $PC^r(D, R; S)$, r∈N. In the sequel, all functional inequalities that we write are assumed to hold finally, that is, for all sufficiently large t.

Let $\gamma = \max\{\tau_j, \sigma_i\}$ and let $t_1 \ge t_0$. By a solution of equation (1.1), we mean a function $x(t) \in PC[[t_1 - \gamma, \infty), R]$ such that $x(t) + \sum_{j=1}^{M} p_j(t)x(t - \tau_j(t))$ is piece-wise continuously differentiable for $t \ge t_1$ and such that equation (1.1) is satisfied for all $t \ge t_1$.

Let $t_1 \ge t_0$ be a given initial point and let $\varphi \in PC[[t_1 - \gamma, t_1], R]$ be a given initial function. Then if $x(t - \tau_j(t)) \in PC^1[[t_1 - \gamma, t_1], R]$, $x(t - \sigma_i(t)) \in PC[[t_1 - \gamma, t_1], R]$ and

$$\lim_{t\to\infty}(t - \tau_j(t)) = \lim_{t\to\infty}(t - \sigma_i(t)) = \infty, \ \forall t \ge t_1, \ \tau_j, \ \sigma_i \in R_+, \qquad (1.4)$$

equation (1.1) has a unique solution on $[t_1, \infty)$ satisfying the initial condition

$$x(t) = \varphi(t) \text{ for } t_1 - \gamma \le t \le t_1 \qquad (1.5)$$

({Bainov/Simeonov: 1998}).

A solution x of the initial value problem (1.1) and (1.5) on $[t_1, \infty)$ is said to be

(i) finally positive, if there exists $T \ge 0$ such that $x(t)$ is defined for $t \ge T$ and $x(t) > 0$ for all $t \ge T$;

(ii) finally negative, if there exists $T \ge 0$ such that $x(t)$ is defined for $t \ge T$ and $x(t) < 0$ for all $t \ge T$;

(iii) regular, if it is defined in some half line $[T_x, \infty)$ for some $T_x \in R$ and $\sup\{|x(t)| : t \ge T\} > 0, \ \forall T > T_x$ ({Lakshmikantham et al: 1989}).

In what follows, we will denote by $x(\varphi; t)$, the unique solution of the initial value problem (1.1) and (1.5). It exists throughout the interval $[t_1 - \gamma, +\infty)$.

2. Statement of the Problem

The comparison results for the solutions of non-neutral delay impulsive differential equations and inequalities have been studied by many authors [{Kulenovic et al: 1990}; {Ladas et al: 1992}; {Agarwal et al: 1996}; {Belinskiy et al: 2007}; {Berezanski/Braverman: 2002}; {Domshlak et al: 2002}; {El-Morshedy/Grace: 2005}]. A comprehensive treatment of such results is given in the monograph by Bainov and Simeonov ({Bainov/Simeonov: 1998}). One of the most important methods of such investigations is the method of generalized characteristic equations, which is based on the idea of finding the solutions of linear impulsive systems of the form

$$x(t)=\varphi(t_0)\exp\left(-\int_{t_0}^{t}\alpha(s)ds\right)\prod_{t_0\le t_\ell<t}(1-\alpha_\ell),\ t\ge t_0. \tag{2.1}$$

Our main objective is to apply this method to equation (1.1) and inequalities (1.2) and (1.3) to find comparison conditions and to generalize and extend the idea for the determination of the oscillatory properties earlier referred to.

We return to equation (1.1) and based on it, introduce the conditions

C2.1 $$\begin{cases} p_j\in PC^1([t_0,\infty),R_+),\ \tau_j\in C^1([t_0,\infty),R_+);\ 1\le j\le M,\\ q_i\in PC([t_0,\infty),R_+),\ \sigma_i\in C([t_0,\infty),R_+);\ 1\le i\le N\end{cases}$$

and

C2.2 Let $t_0\in R_+$ and define $t_{-1}=\min\{T_{-1},\ \tilde{T}_{-1}\}$, where $T_{-1}=\min\limits_{1\le j\le M}\left\{\inf\limits_{t\ge t_0}\{t-\tau_j(t)\}\right\}$ and $\tilde{T}_{-1}=\min\limits_{1\le i\le N}\left\{\inf\limits_{t\ge t_0}\{t-\sigma_i(t)\}\right\}$.

For each $t_0\in R_+$, we define the following functions:

$$\begin{cases} h_j(t)=\min\limits_{t\ge t_0}\{t_0,t-\tau_j(t)\},\quad H_j(t)=\max\limits_{t\ge t_0}\{t_0,t-\tau_j(t)\};\quad 1\le j\le M\\ g_i(t)=\min\limits_{t\ge t_0}\{t_0,t-\sigma_i(t)\},\quad G_i(t)=\max\limits_{t\ge t_0}\{t_0,t-\sigma_i(t)\};\quad 1\le i\le M.\end{cases} \tag{2.2}$$

and write the generalized characteristic system as

$$\begin{cases}\alpha(t)=S_{pq}[\alpha](t)\\ \alpha_k=I_{pq}[\alpha](k),\end{cases} \tag{2.3}$$

where $t\ge t_0$, $t_k\ge t_0$ and $t_k\in S$ for all $k\in Z$; S_{pq}, I_{pq} are generalized characteristic operators acting on the neutral delay impulsive differential equations with the coefficients p and q for t and t_k respectively; and

$$\begin{cases} S_{pq}[\alpha](t)=\Lambda_{11}(t)\prod\limits_{H_j(t)\le t_\ell<t}(1-\alpha_\ell)^{-1}+\Lambda_{21}(t)\prod\limits_{G_i(t)\le t_\ell<t}(1-\alpha_\ell)^{-1},\\ I_{pq}[\alpha](k)=\Lambda_{12}(t_k)\prod\limits_{H_j(t_k)\le t_\ell<t_k}(1-\alpha_\ell)^{-1}+\Lambda_{22}(t_k)\prod\limits_{G_i(t_k)\le t_\ell<t_k}(1-\alpha_\ell)^{-1}.\end{cases} \tag{2.4}$$

The operator elements $\Lambda_{m'n'}(*)$, $1\le m',\ n'\le 2$, are defined by

$$\Lambda_{11}(t)=\sum_{j=1}^{M}\left[p_j(t)(\tau_j'(t)-1)\frac{\varphi'(h_j(t))}{\varphi'(t_0)}\alpha(H_j(t))+p_j'(t)\frac{\varphi(h_j(t))}{\varphi(t_0)}\right]\exp\left(\int_{H_j(t)}^{t}\alpha(s)ds\right),$$

$$\Lambda_{12}(t_k)=\sum_{j=1}^{M}\left[p_j(t_k)(\Delta\tau_j(t_k)-1)\frac{\Delta\varphi(h_j(t_k))}{\Delta\varphi(t_0)}\alpha(H_j(t_k))+\Delta p_j(t_k)\frac{\varphi(h_j(t_k))}{\varphi(t_0)}\right]\exp\left(\int_{H_j(t_k)}^{t_k}\alpha(s)ds\right),$$

$$\Lambda_{21}(t)=\sum_{i=1}^{N}q_i(t)\frac{\varphi(g_i(t))}{\varphi(t_0)}\exp\left(\int_{G_i(t)}^{t}\alpha(s)ds\right)$$

and

$$\Lambda_{22}(t_k)=\sum_{i=1}^{N}q_{ik}\frac{\varphi(g_i(t_k))}{\varphi(t_0)}\exp\left(\int_{G_i(t_k)}^{t_k}\alpha(s)ds\right),\ \forall t>t_0.$$

Let $J\subset R$ be a given interval and $\Omega(J)$ a set of piece-wise continuous functions defined in J. In future, the notation $(\alpha,\ \{\alpha_k\}_{k=1}^{\infty})\in\Omega(J)$, where $\alpha(t)$ is the value of α at t, will mean that the function α and the sequence $\{\alpha_k\}_{k=1}^{\infty}$ satisfy the condition

$\alpha\in PC(J,R)$, $\alpha_k<1$ $\forall k\in Z$ such that $t_k\in[t_0,\infty)\cap S$.

Now let equation (1.4) and condition

C2.3 $\begin{cases} \tau_j \in C^1(R_+, R_+), \ \sigma_i \in C(R_+, R_+), \\ \lim\limits_{t\to+\infty}(t-\tau_j(t)) = \lim\limits_{t\to+\infty}(t-\sigma_i(t)) = \infty, \quad 1 \le j \le M \text{ and } 1 \le i \le N \end{cases}$

be fulfilled. Further, let the coefficients of equation (1.1) be non-negative, that is,

C2.4 $\begin{cases} p_j \in PC^1([t_0, \infty), R_+), q_i \in PC([t_0, \infty), R_+), \\ q_{ik} \ge 0 \text{ for } k \in N, \ 1 \le j \le M \text{ and } 1 \le i \le N. \end{cases}$

Define

$$\overline{M} = \max\{M, N\},$$

and let $x(t)$ be a finally positive solution of equation (1.1).

Remark 2.1 (Equivalent of Remark 16.2 by Bainov and Simeonov ({Bainov/Simeonov: 1998})

Assume that condition C2.3 holds. Then for sufficiently large t, $h_j(t) = t_0$, $H_j(t) = t - \tau_j(t)$, $g_i(t) = t_0$ and $G_i(t) = t - \sigma_i(t)$ for $1 \le j \le M$, $1 \le i \le N$ and $\forall t \in [t_0, T)\setminus S$ system (2.3) transforms into the equation

$$\begin{cases} \alpha(t) = \sum\limits_{j=1}^{M}\left[p_j(t)(\tau_j'(t)-1)\alpha(t-\tau_j(t)) + p_j'(t)\right]\exp\left(\int\limits_{t-\tau_j(t)}^{t}\alpha(s)ds\right)\prod\limits_{t-\tau_j(t)\le t_\ell<t}(1-\alpha_\ell)^{-1} + \\ \qquad + \sum\limits_{i=1}^{N} q_i(t)\exp\left(\int\limits_{t-\sigma_i(t)}^{t}\alpha(s)ds\right)\prod\limits_{t-\sigma_i(t)\le t_\ell<t}(1-\alpha_\ell)^{-1}, \\ \alpha_k = \sum\limits_{j=1}^{M}\left[p_j(t_k)(\Delta\tau_j(t_k)-1)\alpha(t_k-\tau_j(t_k)) + \Delta p_j(t_k)\right]\exp\left(\int\limits_{t_k-\tau_j(t_k)}^{t_k}\alpha(s)ds\right)\prod\limits_{t_k-\tau_j(t_k)\le t_\ell<t_k}(1-\alpha_\ell)^{-1} + \\ \qquad + \sum\limits_{i=1}^{N} q_{ik}\exp\left(\int\limits_{t_k-\sigma_i(t_k)}^{t_k}\alpha(s)ds\right)\prod\limits_{t_k-\sigma_i(t_k)\le t_\ell<t_k}(1-\alpha_\ell)^{-1}. \end{cases} \tag{2.5}$$

Corollary 2.1 (Equivalent of Corollary 16.2 by Bainov and Simeonov ({Bainov/Simeonov: 1998})

Let conditions C2.1 and C2.3 be fulfilled. Suppose equation (1.1) has a finally positive solution x(t) for $t \in [t_0, T)\setminus S$ and $x(t_0^+) > 0$ $\forall k \in N$ such that $t_k \in S$. Then there exist $t_1 > 0$ and $(\alpha, \{\alpha_k\}_{k=1}^{\infty}) \in \Omega([t_{-1}, +\infty))$ such that $(\alpha, \{\alpha_k\}_{k=1}^{\infty})$ is a solution of equation (2.4) for $t > t_1 \ge 0$.

Now, let condition C2.3 be fulfilled and suppose that the coefficients of equation (1.1) are nonnegative, that is, condition C2.4 is satisfied. Let x be a finally positive solution of equation (1.1), and let the function α and the sequence $\{\alpha_k\}_{k=1}^{\infty}$ be defined as

$$\alpha(t) = -\frac{x'(t)}{x(t)}, \quad \alpha_k = -\frac{\Delta x(t_k)}{x(t_k)} \tag{2.6}$$

for $t \in [t_0, T)\setminus S$ and $\forall k \in N$, $t_k \in S$. Then from Corollary 2.1, there exists $t_1 > 0$ such that $(\alpha, \{\alpha_k\}_{k=1}^{\infty})$ is a solution of equation (2.5) for $t \ge t_0$ and what is more,

$$\sum_{\xi=1}^{\overline{M}}\left[p_\xi'(t) + q_\xi(t)\right] \le \alpha(t), \ \sum_{\xi=1}^{\overline{M}}\left[\Delta p_\xi(t_k) + q_{\xi k}(t)\right] \le \alpha_k < 1, \ x(t_k^+) > 0 \tag{2.7}$$

for $t \ge t_1$, $t_k \ge t_1$, $1 \le k < \infty$, $1 \le \xi \le \overline{M}$.

Let t_{-1} be as defined in condition C2.2, $J := [t_{-1}, +\infty)$ and set

$$\begin{cases} \tilde{S}_{pq}[\lambda](t) = \sum_{j=1}^{M}\left[p_j(t)\left(\tau_j'(t)-1\right)\lambda(t-\tau_j(t)) + p_j'(t)\right]\exp\left(\int_{t-\tau_j(t)}^{t}\lambda(s)ds\right)\prod_{t-\tau_j(t)\le t_\ell<t}(1-\lambda_\ell)^{-1} + \\ \quad + \sum_{i=1}^{N} q_i(t)\exp\left(\int_{t-\sigma_i(t)}^{t}\lambda(s)ds\right)\prod_{t-\sigma_i(t)\le t_\ell<t}(1-\lambda_\ell)^{-1}, \\ \tilde{I}_{pq}[\lambda](k) = \sum_{j=1}^{M}\left[p_j(t_k)\left(\Delta\tau_j(t_k)-1\right)\alpha(t_k-\tau_j(t_k)) + \Delta p_j(t_k)\right]\exp\left(\int_{t_k-\tau_j(t_k)}^{t_k}\alpha(s)ds\right)\prod_{t_k-\tau_j(t_k)\le t_\ell<t_k}(1-\alpha_\ell)^{-1} + \\ \quad + \sum_{i=1}^{N} q_{ik}\exp\left(\int_{t_k-\sigma_i(t_k)}^{t_k}\alpha(s)ds\right)\prod_{t_k-\sigma_i(t_k)\le t_\ell<t_k}(1-\alpha_\ell)^{-1} \end{cases} \tag{2.8}$$

for $\left(\lambda,\ \{\lambda_k\}_{k=1}^{\infty}\right)\in\Omega(J)$ and $t\ge t_1$, $t_k\ge t_1$, $1\le k<\infty$. Then:

(i) $\tilde{S}_{pq}[0](t)=\sum_{\xi=1}^{\bar{M}}\left[p_\xi'(t)+q_\xi(t)\right]$, $\tilde{I}_{pq}[0](k)=\sum_{\xi=1}^{\bar{M}}\left[\Delta p_\xi(t_k)+q_{\xi k}(t)\right]$, $t\ge t_1$, $t_k\ge t_1$ and $t_k\in S$;

(ii) if $\left(\beta,\ \{\beta_k\}_{k=1}^{\infty}\right)\in\Omega(J)$, $\left(\gamma,\ \{\gamma_k\}_{k=1}^{\infty}\right)\in\Omega(J)$ and

$$\beta(t)\le\gamma(t),\quad \beta_k\le\gamma_k,\quad t\ge t_{-1},\ k\in N$$

then

$$\begin{cases} \tilde{S}_{pq}[\beta](t)\le\tilde{S}_{pq}[\gamma](t),\ t\notin S, \\ \tilde{I}_{pq}[\beta](k)\le\tilde{I}_{pq}[\gamma](k),\ \forall t_k\in S. \end{cases}$$

To each $t_1>0$ and t_{-1} as defined in condition C2.2, we relay the following inductively defined functional and numerical sequences:

$$\begin{cases} u_0(t)\equiv 0,\ \text{for}\ t\ge t_{-1} \\ u_{0k}\equiv 0,\ \text{for}\ t_k\ge t_{-1} \end{cases}$$

$$\begin{cases} u_{m+1}(t)=\begin{cases} 0,\ t_{-1}\le t<t_1 \\ \tilde{S}_{pq}[u_m](t),\ t\ge t_1, \end{cases} \\ u_{m+1,k}=\begin{cases} 0,\ t_{-1}\le t_k<t_1,\ t_k\in S \\ \tilde{I}_{pq}[u_m](k),\ t_k\ge t_1,\ t_k\in S. \end{cases} \end{cases} \tag{2.9}$$

Theorem 2.1 (Equivalent of Theorem 16.2 by Bainov and Simeonov ({Bainov/Simeonov: 1998})
Let conditions C2.3 and C2.4 hold. Then the following statements are equivalent:

(a) equation (1.1) has a finally positive solution;

(b) there exists $t_1\ge 0$ such that the sequences $\{u_m(t)\}$ and $\{u_{mk}\}$, defined by equation (2.9) converge point-wise and monotonically, that is,

$$\begin{cases} \lim_{m\to+\infty} u_m(t)=u(t),\quad t\ge t_1 \\ \lim_{m\to+\infty} u_{mk}=u_k,\quad t_k\ge t_1,\ t_k\in S \end{cases} \tag{2.10}$$

and

$$u_k<1\ \text{for all}\ t_k\ge t_1\ \text{and}\ t_k\in S. \tag{2.11}$$

3. Main Results

Let $t_0\ge 0$ and t_{-1} be as defined in condition C2.2. We recall the neutral delay impulsive differential equation (1.1) and the inequalities (1.2) and (1.3) and introduce the conditions

C3.1 $$\begin{cases} p_j,\ a_j,\ c_j\in PC^1(R_+,\ R_+),\ q_i,\ b_i,\ d_i\in PC(R_+,\ R_+),\ \tau_j\in C^1(R_+,R_+) \\ \sigma_i\in C(R_+,R_+),\ q_{ik},\ b_{ik},\ d_{ik}\ge 0,\quad k\in N,\quad 1\le j\le M\ \text{and}\ 1\le i\le N, \end{cases}$$

and

C3.2 $\begin{cases} a_j(t) \geq p_j(t) \geq c_j(t);\ b_i(t) \geq q_i(t) \geq d_i(t),\ \forall t \in R_+, \\ b_{ik} \geq q_{ik} \geq d_{ik},\quad k \in N,\quad 1 \leq j \leq M \text{ and } 1 \leq i \leq N. \end{cases}$

Also, let us now suppose that the initial interval associated with the said equation and inequalities under the above conditions is $[t_{-1}, t_0]$.

Theorem 3.1

Let conditions C3.1 and C3.2 be fulfilled. Assume that x, y and z belong to the space $PC([t_{-1}, +\infty), R)$ and are solutions of equation (1.1) and inequalities (1.2) and (1.3) respectively. Suppose further, that

$$y(t) > 0,\ t \geq t_0, \tag{3.1}$$

$$z(t_0^+) \geq x(t_0^+) \geq y(t_0^+), \tag{3.2}$$

$$\frac{y(t)}{y(t_0)} \geq \frac{x(t)}{x(t_0)} \geq \frac{z(t)}{z(t_0)} \geq 0,\ t_{-1} \leq t \leq t_0. \tag{3.3}$$

Then

$$z(t) \geq x(t) \geq y(t),\ \forall t \geq t_0. \tag{3.4}$$

Proof

It follows from inequalities (3.1) and (3.2) that the functions x and z are positively defined in some right neighbourhood $[t_0, T_1)$ of t_0. We shall show that $T_1 = +\infty$. If we assume on the contrary, then there exists $T_1 > t_0$ such that

$$x(t) > 0,\ z(t) > 0,\ t_0 \leq t < T_1 \tag{3.5}$$

and

$$x(T_1^+) \leq 0 \ \text{ or } \ z(T_1^+) \leq 0. \tag{3.6}$$

For $t \in [t_0, T_1)$ and $t_k \in [t_0, T_1) \cap S$, we set

$$\begin{cases} \alpha_0(t) = -\dfrac{y'(t)}{y(t)},\ \beta_0(t) = -\dfrac{x'(t)}{x(t)},\ \gamma_0(t) = -\dfrac{z'(t)}{z(t)}, \\ \alpha_{0k} = -\dfrac{\Delta y(t_k)}{y(t_k)},\ \beta_{0k} = -\dfrac{\Delta x(t_k)}{x(t_k)},\ \gamma_{0k} = -\dfrac{\Delta z(t_k)}{z(t_k)}, \end{cases}$$

then

$$\alpha_0(t) \geq S_{ab}[\alpha_0](t),\ \alpha_{0k} \geq I_{ab}[\alpha_0](k), \tag{3.7}$$

$$\beta_0(t) = S_{pq}[\beta_0](t),\ \beta_{0k} = I_{pq}[\beta_0](k), \tag{3.8}$$

$$\gamma_0(t) \leq S_{cd}[\gamma_0](t),\ \gamma_{0k} \leq I_{cd}[\gamma_0](k) \tag{3.9}$$

for $t \in [t_0, T_1) \setminus S,\ t_k \in S$, where

$$\begin{cases} S_{ab}[\alpha](t) = \displaystyle\sum_{j=1}^{M} \left[a_j(t)(\tau_j'(t) - 1) \frac{\varphi'(h_j(t))}{\varphi'((t_0))} \alpha(H_j(t)) + a_j'(t) \frac{\varphi(h_j(t))}{\varphi((t_0))} \right] x \\ \qquad x \exp\left(\displaystyle\int_{H_j(t)}^{t} \alpha(s) ds \right) \prod_{H_j(t) \leq t_\ell < t} (1 - \alpha_\ell)^{-1} + \sum_{i=1}^{N} b_i(t) \frac{\varphi(g_i(t))}{\varphi((t_0))} \exp\left(\int_{G_i(t)}^{t} \alpha(s) ds \right) \prod_{G_i(t) \leq t_\ell < t} (1 - \alpha_\ell)^{-1} \\ I_{ab}[\alpha](k) = \displaystyle\sum_{j=1}^{M} \left[a_j(t_k)(\Delta \tau_j(t) - 1) \frac{\Delta\varphi(h_j(t))}{\Delta\varphi((t_0))} \alpha(H_j(t_k)) + \Delta a_j(t) \frac{\varphi(h_j(t_k))}{\varphi((t_0))} \right] x \\ \qquad x \exp\left(\displaystyle\int_{H_j(t_k)}^{t_k} \alpha(s) ds \right) \prod_{H_j(t_k) \leq t_\ell < t_k} (1 - \alpha_\ell)^{-1} + \sum_{i=1}^{N} b_{ik} \frac{\varphi(g_i(t_k))}{\varphi((t_0))} \exp\left(\int_{G_i(t_k)}^{t} \alpha(s) ds \right) \prod_{G_i(t_k) \leq t_\ell < t_k} (1 - \alpha_\ell)^{-1} \end{cases}$$

$$\begin{cases} S_{pq}[\beta](t) = \sum_{j=1}^{M}\left[p_j(t)(\tau_j'(t)-1)\dfrac{\varphi'(h_j(t))}{\varphi'((t_0))}\beta(H_j(t)) + p_j'(t)\dfrac{\varphi(h_j(t))}{\varphi((t_0))}\right]x \\ \qquad x\exp\left(\int_{H_j(t)}^{t}\beta(s)ds\right)\prod_{H_j(t)\le t_\ell<t}(1-\beta_\ell)^{-1} + \sum_{i=1}^{N} q_i(t)\dfrac{\varphi(g_i(t))}{\varphi((t_0))}\exp\left(\int_{G_i(t)}^{t}\beta(s)ds\right)\prod_{G_i(t)\le t_\ell<t}(1-\beta_\ell)^{-1} \\ I_{pq}[\beta](k) = \sum_{j=1}^{M}\left[p_j(t_k)(\Delta\tau_j(t)-1)\dfrac{\Delta\varphi(h_j(t))}{\Delta\varphi((t_0))}\beta(H_j(t_k)) + \Delta p_j(t)\dfrac{\varphi(h_j(t_k))}{\varphi((t_0))}\right]x \\ \qquad x\exp\left(\int_{H_j(t_k)}^{t_k}\beta(s)ds\right)\prod_{H_j(t_k)\le t_\ell<t_k}(1-\beta_\ell)^{-1} + \sum_{i=1}^{N} q_{ik}\dfrac{\varphi(g_i(t_k))}{\varphi((t_0))}\exp\left(\int_{G_i(t_k)}^{t}\beta(s)ds\right)\prod_{G_i(t_k)\le t_\ell<t_k}(1-\beta_\ell)^{-1} \end{cases}$$

$$\begin{cases} S_{cd}[\gamma](t) = \sum_{j=1}^{M}\left[c_j(t)(\tau_j'(t)-1)\dfrac{\varphi'(h_j(t))}{\varphi'((t_0))}\gamma(H_j(t)) + c_j'(t)\dfrac{\varphi(h_j(t))}{\varphi((t_0))}\right]x \\ \qquad x\exp\left(\int_{H_j(t)}^{t}\gamma(s)ds\right)\prod_{H_j(t)\le t_\ell<t}(1-\gamma_\ell)^{-1} + \sum_{i=1}^{N} d_i(t)\dfrac{\varphi(g_i(t))}{\varphi((t_0))}\exp\left(\int_{G_i(t)}^{t}\gamma(s)ds\right)\prod_{G_i(t)\le t_\ell<t}(1-\gamma_\ell)^{-1} \\ I_{cd}[\gamma](k) = \sum_{j=1}^{M}\left[c_j(t_k)(\Delta\tau_j(t)-1)\dfrac{\Delta\varphi(h_j(t))}{\Delta\varphi((t_0))}\gamma(H_j(t_k)) + \Delta c_j(t)\dfrac{\varphi(h_j(t_k))}{\varphi((t_0))}\right]x \\ \qquad x\exp\left(\int_{H_j(t_k)}^{t_k}\gamma(s)ds\right)\prod_{H_j(t_k)\le t_\ell<t_k}(1-\gamma_\ell)^{-1} + \sum_{i=1}^{N} d_{ik}\dfrac{\varphi(g_i(t_k))}{\varphi((t_0))}\exp\left(\int_{G_i(t_k)}^{t}\gamma(s)ds\right)\prod_{G_i(t_k)\le t_\ell<t_k}(1-\gamma_\ell)^{-1} \end{cases}$$

and the functions h_j, H_j, g_i and G_i are as defined in the functions (2.2).

We shall prove that

$$\begin{cases} \alpha_0(t) \ge \beta_0(t) \ge \gamma_0(t),\ t_0 \le t < T_1 \\ \alpha_{0k} \ge \beta_{0k} \ge \gamma_{0k},\ t_0 \le t_k < T_1, t_k \in S. \end{cases} \tag{3.10}$$

However, in doing this, we shall only restrict ourselves to the proof of

$$\alpha_0(t) \ge \beta_0(t),\ \alpha_{0k} \ge \beta_{0k},\ t_0 \le t, t_k < T_1 \text{ and } t_k \in S, \tag{3.11}$$

since the proof of

$$\beta_0(t) \ge \gamma_0(t),\ \beta_{0k} \ge \gamma_{0k},\ t_0 \le t, t_k < T_1 \text{ and } t_k \in S$$

is analogous and is omitted.

It follows from conditions (3.1) and (3.5) that $(\alpha_0, \{\alpha_{0k}\}_{k=1}^{\infty})$, $(\beta_0, \{\beta_{0k}\}_{k=1}^{\infty})$ and $(\gamma_0, \{\gamma_{0k}\}_{k=1}^{\infty})$ belong to $\Omega([t_0, T_1))$, that is, $\alpha_0, \beta_0, \gamma_0 \in PC([t_0, T_1), R)$ and $\alpha_{0k}, \beta_{0k}, \gamma_{0k} < 1$ for $\forall k : t_k \in [t_0, T_1) \cap S$.

On the other hand, equation (3.8) suggests that $(\beta_0, \{\beta_{0k}\}_{k=1}^{\infty})$ is a solution of the system

$$\begin{cases} \beta(t) = S_{pq}[\beta](t) \\ \beta_k = I_{pq}[\beta](k) \end{cases} \tag{3.12}$$

$\forall t \in [t_0, T_1)$. Therefore,

$$\begin{cases} \beta_0(t) \equiv \beta(t) \le \alpha_0(t),\quad t \in [t_0, T_1), \\ \beta_{0k} \equiv \beta_k \le \alpha_{0k},\quad t_k \in [t_0, T_1) \cap S, \end{cases} \tag{3.13}$$

and this is a proof of inequalities (3.11).

Obviously,

$$y(t) = y(T_0^+)\exp\left(-\int_{t_0}^{t}\alpha_0(s)ds\right)\prod_{t_0<t_k<t}(1-\alpha_{0k}),$$

$$x(t) = x(T_0^+)\exp\left(-\int_{t_0}^{t}\beta_0(s)ds\right)\prod_{t_0<t_k<t}(1-\beta_{0k}),$$

$$z(t) = z(T_0^+)\exp\left(-\int_{t_0}^{t}\gamma_0(s)ds\right)\prod_{t_0<t_k<t}(1-\gamma_{0k}).$$

Therefore, from inequalities (3.2) and (3.10) it follows that

$$z(t) \geq x(t) \geq y(t), \quad t_0 \leq t < T_1. \tag{3.14}$$

Hence, from inequality (3.14) and the left-continuity of the functions y, x and z,

$$z(T_1) \geq x(T_1) \geq y(T_1) > 0,$$

which is contradictory to inequalities (3.6) for $T_1 \notin S$.

If $T_1 = t_m$ for some $m \in N$, then equation (3.8) and inequalities (3.7), (3.9) and (3.10) suggest that

$$1 > \alpha_{om} \geq \beta_{0m} \geq \gamma_{0m}.$$

This implies that $x(T_1) > 0$ and $z(T_1) > 0$, which further confirms the contradiction of inequalities (3.6). Consequently, relation (3.4) holds for each $t \geq t_0$. This completes the proof of Theorem 3.1.

The following are some corollaries arising from the proved theorem and are generally important, especially in establishing linearized oscillation theorems for neutral impulsive equations.

Corollary 3.1

Let the following conditions hold:

C3.3 $$\begin{cases} a_j \in PC^1(R_+, R_+),\ b_i \in PC(R_+, R_+),\ \tau_j \in C^1(R_+,\ R_+), \\ \sigma_i \in C(R_+,\ R_+),\ b_{ik} \geq 0,\quad k \in N,\quad 1 \leq j \leq M \ \text{and}\ 1 \leq i \leq N, \end{cases}$$

Let $t_0 \geq 0$ and $\varphi,\ \psi \in PC([t_{-1}, t_0],\ R)$ be such that

$$\psi(t_0) \geq \varphi(t_0) > 0, \frac{\varphi(t)}{\varphi(t_0)} \geq \frac{\psi(t)}{\psi(t_0)} \geq 0, \quad t_{-1} \leq t \leq t_0 \tag{3.15}$$

and

$$x(\varphi;\ t) > 0,\ t \geq t_0.$$

Then

$$x(\psi; t) \geq x(\varphi; t),\ t \geq t_0.$$

Now consider the neutral impulsive differential inequality (1.2) together with the equation

$$\begin{cases} \left[y(t) + \sum_{j=1}^{M} a_j(t)y(t-\tau_j(t))\right]' + \sum_{i=1}^{N} b_i(t)y(t-\sigma_i(t)) = 0, t \notin S \\ \Delta\left[y(t_k) + \sum_{j=1}^{M} a_j(t_k)y(t_k-\tau_j(t_k))\right] + \sum_{i=1}^{N} b_{ik}y(t_k-\sigma_i(t_k)) = 0, \forall t_k \in S. \end{cases} \tag{3.16}$$

From Theorem 3.1, we immediately obtain the following result.

Corollary 3.2

Let conditions C3.3 be fulfilled. Then the following statements are equivalent:

(a) The neutral impulsive inequality (1.2) has a finally positive solution;

(b) Equation (3.16) has a finally positive solution.

Comparison results can also be extended to impulsive differential equations and inequalities with advanced arguments by way of adapting the proofs from the present section. For example, the following results are analogous to Corollary 3.2.

Corollary 3.3

Let conditions C3.3 be fulfilled. Then the following statements are equivalent:

(a) The inequality

$$\begin{cases}\left[x(t)+\sum_{j=1}^{M}p_j(t)x(t-\tau_j(t))\right]' -\sum_{i=1}^{N}q_i(t)x(t-\sigma_i(t))\geq 0,\ t\notin S\\ \Delta\left[x(t_k)+\sum_{j=1}^{M}p_j(t_k)x(t_k-\tau_j(t_k))\right]-\sum_{i=1}^{N}q_{ik}x(t_k-\sigma_i(t_k))\geq 0,\ \forall t_k\in S\end{cases}$$

has a finally positive solution

(b) The equation

$$\begin{cases}\left[x(t)+\sum_{j=1}^{M}p_j(t)x(t-\tau_j(t))\right]' -\sum_{i=1}^{N}q_i(t)x(t-\sigma_i(t))= 0,\ t\notin S\\ \Delta\left[x(t_k)+\sum_{j=1}^{M}p_j(t_k)x(t_k-\tau_j(t_k))\right]-\sum_{i=1}^{N}q_{ik}x(t_k-\sigma_i(t_k))= 0,\ \forall t_k\in S\end{cases}$$

has a finally positive solution.

Let us now return to the neutral delay impulsive differential equation (3.16) and together with it, consider the equation

$$\begin{cases}\left[y(t)+\sum_{j=1}^{M}p_j(t)y(t-\mu_j(t))\right]' +\sum_{i=1}^{N}q_i(t)y(t-\lambda_i(t))= 0,\ t\notin S\\ \Delta\left[y(t_k)+\sum_{j=1}^{M}p_j(t_k)y(t_k-\mu_j(t_k))\right]+\sum_{i=1}^{N}q_{ik}\,y(t_k-\lambda_i(t_k))= 0,\ \forall t_k\in S.\end{cases}\tag{3.17}$$

We introduce the following conditions:

C3.4
$$\begin{cases}a_j\in PC^1(R_+,R_+),\ b_i\in PC(R_+,R_+),\ \tau_j\in C^1(R_+,\ R_+),\sigma_i\in C(R_+,\ R_+),\ b_{ik}\geq 0,\ k\in N,\\ \lim_{t\to+\infty}[t-\tau_j(t)]=\lim_{t\to+\infty}[t-\sigma_i(t)]=+\infty,\ 1\leq j\leq M,\ 1\leq i\leq N\end{cases}\tag{3.18}$$

and

C3.5
$$\begin{cases}p_j\in PC^1(R_+,R_+),\ q_i\in PC(R_+,R_+),\ \mu_j\in C^1(R_+,\ R_+),\lambda_i\in C(R_+,\ R_+),\ q_{ik}\geq 0,\ k\in N,\\ \lim_{t\to+\infty}[t-\mu_j(t)]=\lim_{t\to+\infty}[t-\lambda_i(t)]=+\infty,\ 1\leq j\leq M,\ 1\leq i\leq N.\end{cases}\tag{3.19}$$

Theorem 3.2

Suppose conditions C3.4 and C3.5 are fulfilled. Let, for each $1\leq j\leq M$, $1\leq i\leq N$,

$$\begin{cases}a_j(t)\geq p_j(t),\ b_i(t)\geq q_i(t),\ b_{ik}\geq q_{ik},\ \tau_j(t)\geq\mu_j(t),\\ \sigma_i(t)\geq\lambda_i(t),\ t\in R_+\ \forall k\in N.\end{cases}\tag{3.20}$$

Suppose further, that each regular solution of equation (3.17) is oscillatory. Then each regular solution of equation (3.16) is oscillatory.

Proof

For each $t_0\geq 0$, we set $t_{-1}=\min\{T_{-1},\tilde{T}_{-1}\}$ and $t^*_{-1}=\min\{T^*_{-1},\tilde{T}^*_{-1}\}$, where

$$T_{-1}=\min_{1\leq j\leq M}\left\{\inf_{t\geq t_0}\{t-\tau_j(t)\}\right\},\ \tilde{T}_{-1}=\min_{1\leq i\leq N}\left\{\inf_{t\geq t_0}\{t-\sigma_i(t)\}\right\}$$

and

$$T^*_{-1}=\min_{1\leq jM}\left\{\inf_{t\geq t_0}\{t-\mu_j(t)\}\right\},\ \tilde{T}^*_{-1}=\min_{1\leq i\in N}\left\{\inf_{t\geq t_0}\{t-\lambda_i(t)\}\right\},$$

and define the sequences $\{v_m(t)\}$ and $\{v_{mk}\}$ as follows:

$v_0(t)\equiv 0$, $v_{0k}\equiv 0$, for $t\geq t_{-1}$, $t_k\geq t_{-1}$ and $t_k\in S$

$$v_{m+1}(t)=\begin{cases}0, & t_{-1}\le t<t_0,\\ \sum_{j=1}^{M}\left[a_j(t)\left(\tau_j'(t)-1\right)v_m\left(t-\tau_j(t)\right)+a_j'(t)\right]\exp\left(\int_{t-\tau_j(t)}^{t}v_m(s)ds\right)\prod_{t-\tau_j(t)\le t_\ell<t}\left(1-v_{m\ell}\right)^{-1}+ & \\ +\sum_{i=1}^{N}b_i(t)\exp\left(\int_{t-\sigma_i(t)}^{t}v_m(s)ds\right)\prod_{t-\sigma_i(t)\le t_\ell<t}\left(1-v_{m\ell}\right)^{-1}, & t\ge t_0\end{cases}$$

$$v_{m+1,k}=\begin{cases}0, & t_{-1}\le t_k<t_0,\forall t_k\in S.\\ \sum_{j=1}^{M}\left[a_j(t_k)\left(\Delta\tau_j(t)-1\right)v_m\left(t_k-\tau_j(t_k)\right)+\Delta a_j(t_k)\right]\exp\left(\int_{t_k-\tau_j(t_k)}^{t_k}v_m(s)ds\right)\prod_{t_k-\tau_j(t_k)\le t_\ell<t_k}\left(1-v_{m\ell}\right)^{-1}+ & \\ +\sum_{i=1}^{N}b_{ik}\exp\left(\int_{t_k-\sigma_i(t_k)}^{t_k}v_m(s)ds\right)\prod_{t_k-\sigma_i(t_k)\le t_\ell<t_k}\left(1-v_{m\ell}\right)^{-1}, & t_k\ge t_0,\forall t_k\in S.\end{cases}$$

Furthermore, we define the sequences $\{u_m(t)\}$ and $\{u_{mk}\}$ as follows:

$u_0(t)\equiv 0$, $u_{0k}\equiv 0$, for $t\ge t_{-1}^*$, $t_k\ge t_{-1}^*$ and $t_k\in S$

$$u_{m+1}(t)=\begin{cases}0, & t_{-1}\le t<t_0,\\ \sum_{j=1}^{M}\left[p_j(t)\left(\mu_j'(t)-1\right)u_m\left(t-\mu_j(t)\right)+p_j'(t)\right]\exp\left(\int_{t-\mu_j(t)}^{t}u_m(s)ds\right)\prod_{t-\mu_j(t)\le t_\ell<t}\left(1-u_{m\ell}\right)^{-1}+ & \\ +\sum_{i=1}^{N}q_i(t)\exp\left(\int_{t-\lambda_i(t)}^{t}u_m(s)ds\right)\prod_{t-\lambda_i(t)\le t_\ell<t}\left(1-u_{m\ell}\right)^{-1}, & t\ge t_0\end{cases}$$

$$u_{m+1,k}=\begin{cases}0, & t_{-1}\le t_k<t_0,\forall t_k\in S.\\ \sum_{j=1}^{M}\left[p_j(t_k)\left(\Delta\mu_j(t)-1\right)u_m\left(t_k-\mu_j(t_k)\right)+\Delta p_j(t_k)\right]\exp\left(\int_{t_k-\mu_j(t_k)}^{t_k}u_m(s)ds\right)\prod_{t_k-\mu_j(t_k)\le t_\ell<t_k}\left(1-u_{m\ell}\right)^{-1}+ & \\ +\sum_{i=1}^{N}q_{ik}\exp\left(\int_{t_k-\lambda_i(t_k)}^{t_k}u_m(s)ds\right)\prod_{t_k-\mu_i(t_k)\le t_\ell<t_k}\left(1-u_{m\ell}\right)^{-1}, & t_k\ge t_0,\forall t_k\in S\end{cases}$$

For purposes of contradiction, let us assume that equation (3.16) has a non-oscillatory solution, which, in view of its linearity, may be finally positive. Consequently, from Theorem 2.1, there exists $t_0\ge 0$ such that the sequences $\{v_m(t)\}$ and $\{v_{mk}\}$ defined by the formulas above, converge point-wise and monotonically as follows:

$$\begin{cases}\lim_{m\to+\infty}v_m(t)=v(t), & t\ge t_0, t\notin S.\\ \lim_{m\to+\infty}v_{mk}=v_k, & t_k\ge t_0,\forall t_k\in S.\end{cases}\tag{3.21}$$

and

$v_k<1$ for $t_k\ge t_0$ and $t_k\in S$.

It follows from the conditions in inequalities (3.20) that

$$u_m(t)\le v_m(t),\ u_{mk}\le v_{mk}.\tag{3.22}$$

for each $t\ge t_0$, $t_k\ge t_0$ and $k\in N$.

On the other hand, since the sequences $\{u_m(t)\}$ and $\{u_{mk}\}$ are non-decreasing in respect of $m \in N$, then inequalities (3.22) imply that the limits

$$\begin{cases} \lim\limits_{m\to+\infty} u_m(t) = u(t), \ t \geq t_0, t \notin S. \\ \lim\limits_{m\to+\infty} u_{mk} = u_k, \ t_k \geq t_0, \forall t_k \in S. \end{cases} \tag{3.23}$$

exist and

$u_k < 1$ for $t_k \geq t_0$ and $t_k \in S$.

Thus, from Theorem 2.1, equation (3.17) has a finally positive solution which is a contradiction. This completes the proof of Theorem 3.2.

REFERENCES

B. Belinskiy, J. R. Graef and S. Petrovic (2007). A nonlinear sturm-picone comparison theorem for dynamic equations on the scales. Int. J. Difference Equations 2, no.1, 25 – 35.

D. D. Bainov and P. S. Simeonov (1998). Oscillation Theory of Impulsive Differential Equations. International Publishers, Orlando.

G. Ladas, C. Qian and J. Yan (1992). A Comparison Result for the Oscillation of Delay Differential Equations. Proceedings of the American Mathematical Society 114, No.4, 939 - 947.

H. El-Morshedy and S. R. Grace (2005). Comparison theorems for second order nonlinear difference equations. J. Math. Anal. Appl. 306, 106 – 121.

L. Berezanski and E. Braverman (2002). Differential equations with several deviating arguments: Sturmian comparison method in oscillatory theory, ii, Electron. J. Differential Equations 2002, no.31, 1 – 18, (electronic).

M. R. S. Kulenovic, G. Ladas and Y. G. Sficas (1990). Comparison results for Oscillations of Delay Equations. Annali di Matematica Pura ed Applicata (IV), CLVI, 1 - 14.

R. P. Agarwal, W. Lian and C. Yeh (1996). Levin’s comparison theorems for nonlinear second order differential equations. Appl. Math. Lett. 9, 29 – 35.

V. Lakshmikantham, D. D. Bainov and P. S. Simeonov (1989). Theory of Impulsive Differential Equations. World Scientific Publishing Company, Singapore.

Y. Domshlak, G. Kvinikadze and I. P. Stavroulakis (2002). Sturmian comparison method: the version for first order neutral differential equations. Math. Ineq. Appl. 5, no.2, 247 – 256.