A Closer Look at the 24 Game

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Introduction

The 24 Game is a commercially available game that is made by Nasco. You are given a card with four digits taken from the digits 1-9. The object is to add, subtract, multiply and/or divide and get a result of 24. The rules state that you must use all four digits on a card and you must use each digit only once. The game sounds easy, but most people find it very challenging. The only mathematics used is at a basic level. Why would a mathematician get so interested in this game?

Here are a few questions about the 24 game that I would like to discuss:

- Are some cards more difficult to solve than others? If so, what makes one card more difficult than another card?
- How many cards could you make using the four digits?
- Which cards are impossible? What does impossible really mean?
- Why is the number 24 used?

Each 24 Game card is assigned a 1-dot 2-dot or 3-dot designation according to difficulty:
“1-Dot” cards are “easy” “2-Dot” cards are “medium” “3-Dot” cards are “tough”

What determines the difficulty level? It appears that the difficulty of the card is determined by several factors.

1-dot cards: Two at a time. Mostly with divisors.

(8-2)x(8-4)
or
(8+8)+(4x2)

2-dot cards: - two strategies. More subtracting and dividing

(9-1)x(7-4)
2 at a time

3x[7+(4/4)]
3 at a time

Most 3-dots use the three at a time strategy. Uses more digits that are not divisors of 24 and also uses more subtracting and dividing. Many 3-dot cards use what I call the "Go Big!" strategy. With this strategy you get a result bigger than 24 and then subtract or divide to get 24. For example, the solution to the card 2368 is (3*(2+8))-6 and the solution to 2588 is (8+5*8)/2. Only three of the 96 2-dot cards use this strategy.

3x[(5x2)-2]

(9x6)-(6x5)

How many possible 24 game cards could you make? You are choosing 4 digits from 1-9 with repetitions to put on a card, but you have to consider rotations and order.

The two cards above are the same, so the number of cards is not 9^4 =6,561. To determine the number of cards we set up a 1-1 correspondence between possible cards and the arrangements of 4 x's and 8 separators.

1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

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| | | xx | | x | x | |

The above arrangement would correspond to the card 4467. So the total number of 24 game cards would be the same as the number of ways to arrange 4 x's and 8 separators? The answer to this problem is $\binom{12}{4} = \frac{12!}{4!8!} = 495$. How many of these cards are possible, that is, have a solution. Well, clearly the card with all 1's is not possible. Let's look at the list of all cards which are not possible. http://reijnhoudt.nl/24game/index.html There are 91 impossible cards. List of all combinations of four digits from 1 to 9 for which it is impossible to form 24. There are 91.

1111	1112	1113	1114	1115	1116	1117	1119	1122	1123	1124	1125	1133
1159	1167	1177	1178	1179	1189	1199	1222	1223	1299	1355	1499	1557
1558	1577	1667	1677	1678	1777	1778	1899	1999	2222	2226	2279	2299
2334	2555	2556	2599	2677	2777	2779	2799	2999	3358	3467	3488	3555
3577	4459	4466	4467	4499	4779	4999	5557	5558	5569	5579	5777	5778
5799	5899	5999	6667	6677	6678	6699	6777	6778	6779	6788	6999	7777
7778	7779	7788	7789	7799	7888	7899	7999	8888	8889	8899	8999	9999

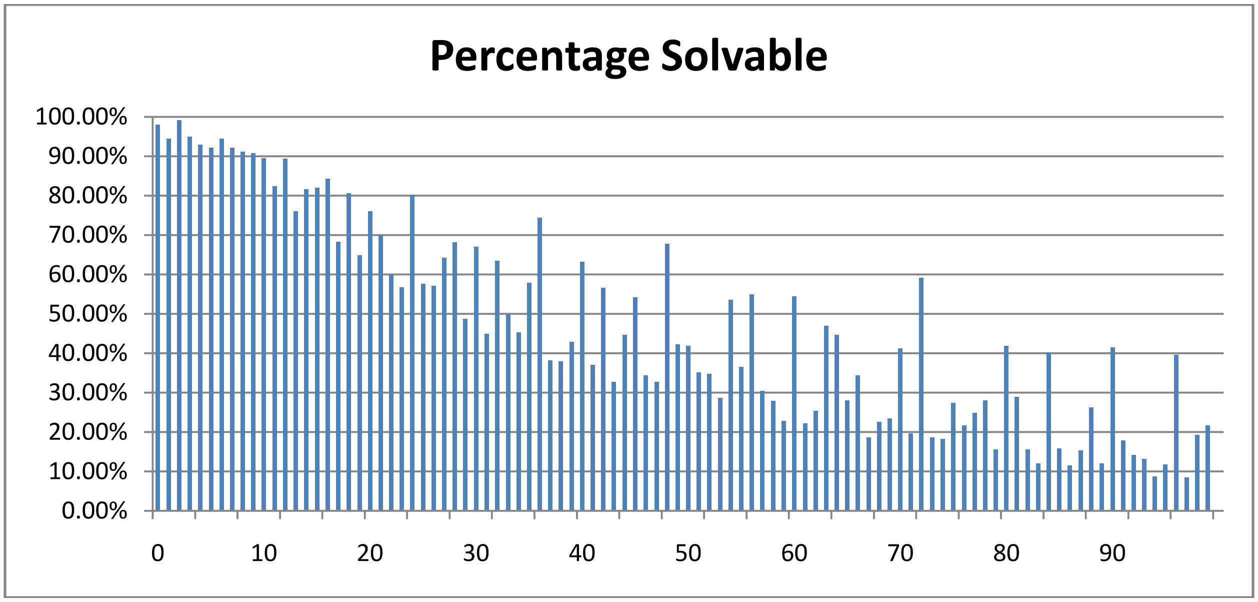
$$8 \div \left[3 - \frac{8}{3}\right]$$
$$= 8 \div \left[\frac{1}{3}\right]$$
$$= 24$$

Consider the card above. I gave it four dots because it cannot be solved staying in the world of integers. Consider the solution: $\frac{8}{3-\frac{8}{3}}$. Should we call the above card solvable? Similar cards are 3388, 3377, 1346, 1456, 1555, 4477, 6618. They are only solved if we are willing to leave the integer world. For a card to be called possible, I will require that the card be solved in the integer world. So why can we buy a box of 24 game cards and not a box of 23 game cards or 48 game cards? The number 23 only has two factors, but the number 48 has several, in fact, even more than 24. When I first started looking at the 24 game I figured that 24 must be maximal in some sense.

Let us determine what percentage of possible 24 game cards are solvable? Let us also determine what percentage of 24 game cards are solvable if we use a different target number? A big thank you to Joern Christian Ploetz for creating the Java program that computed the number of solvable cards for target numbers 0 to 99. Here are our results:

Target Number	# solvable cards	# possible cards	% solvable		Target Number	# solvable cards	# possible cards	% solvable
0	485	495	97.98%		50	207	495	41.82%
1	467	495	94.34%		51	174	495	35.15%
2	491	495	99.19%		52	172	495	34.75%
3	470	495	94.95%		53	142	495	28.69%
4	460	495	92.93%		54	265	495	53.54%
5	456	495	92.12%		55	181	495	36.57%
6	467	495	94.34%		56	272	495	54.95%
7	456	495	92.12%		57	151	495	30.51%
8	451	495	91.11%		58	138	495	27.88%
9	449	495	90.71%		59	113	495	22.83%
10	443	495	89.49%		60	269	495	54.34%
11	408	495	82.42%		61	110	495	22.22%
12	442	495	89.29%		62	125	495	25.25%
13	376	495	75.96%		63	232	495	46.87%
14	404	495	81.62%		64	221	495	44.65%
15	406	495	82.02%		65	139	495	28.08%
16	417	495	84.24%		66	170	495	34.34%
17	338	495	68.28%		67	92	495	18.59%
18	399	495	80.61%		68	112	495	22.63%
19	321	495	64.85%		69	116	495	23.43%
20	376	495	75.96%		70	204	495	41.21%
21	347	495	70.10%		71	97	495	19.60%
22	297	495	60.00%		72	293	495	59.19%
23	281	495	56.77%		73	92	495	18.59%
24	397	495	80.20%		74	90	495	18.18%
25	285	495	57.58%		75	135	495	27.27%
26	283	495	57.17%		76	107	495	21.62%
27	318	495	64.24%		77	123	495	24.85%
28	337	495	68.08%		78	139	495	28.08%
29	241	495	48.69%		79	77	495	15.56%
30	332	495	67.07%		80	207	495	41.82%
31	222	495	44.85%		81	143	495	28.89%
32	314	495	63.43%		82	77	495	15.56%
33	247	495	49.90%		83	59	495	11.92%
34	224	495	45.25%		84	199	495	40.20%
35	286	495	57.78%		85	78	495	15.76%
36	368	495	74.34%		86	57	495	11.52%
37	189	495	38.18%		87	76	495	15.35%
38	188	495	37.98%		88	130	495	26.26%
39	212	495	42.83%		89	59	495	11.92%
40	313	495	63.23%		90	205	495	41.41%
41	183	495	36.97%		91	88	495	17.78%
42	280	495	56.57%		92	70	495	14.14%
43	162	495	32.73%		93	65	495	13.13%
44	221	495	44.65%		94	43	495	8.69%
45	268	495	54.14%		95	58	495	11.72%
46	170	495	34.34%		96	196	495	39.60%
47	162	495	32.73%		97	42	495	8.48%
48	335	495	67.68%		98	95	495	19.19%
49	209	495	42.22%		99	107	495	21.62%

The graph of this is:



I was very surprised that the target number of 24 is not maximal. The top 10 are

Target Number	# cards solvable	% solvable
2	491	99.19%
0	485	97.98%
3	470	94.95%
1	467	94.34%
6	467	94.34%
4	460	92.93%
5	456	92.12%
7	456	92.12%
8	451	91.11%

And the bottom 10 are:

Target Number	# cards solvable	% solvable
97	42	8.48%
94	43	8.69%
86	57	11.52%
95	58	11.72%
83	59	11.92%
89	59	11.92%
93	65	13.13%
92	70	14.14%
87	76	15.35%
79	77	15.56%

The number 24 is actually nineteenth on the list with a percentage of 80.20%. The target number of 18 is slightly better at 80.61%. So why did the manufacturer use 24 as a target?