Theoretical Study of a New Relationship between the Wavelength and Temperature. Compromise with Wien's Law, Radial Parts , and the Emission Spectrum of Hydrogénoid Atoms

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Abstract

Recent theoretical research started on the relationship between the wavelength and temperature, suggests an important compromise. This compromise lies in the constant of proportionality. Wien's law (Established in 1893 by studying the maximum of the emission spectrum of black body) provided a constant of proportionality equal to: $k = 2.89. 10^{-3}$ m ° k, while that given in this study is $k = 9.592. 10^{-3}$ m ° k , which is closer to reality. In addition , a work has been done to study the correlation of the radial part of atomic orbitals and the emission spectrum of hydrogénoid atoms.

Keywords: quantum chemistry, gaz microparticles, Wien's law

Introduction.

The deduction of this relation is simple, it predicts that Einstein photon energy might be equalized to translational energy carried by gaz microparticles which absorb a quantum of energy $\lambda\nu$ and convert it into heat by intramolecular interaction or into electromagnetic radiation emission by rotation or vibration or both of them and having the same Einstein's photons magnitude.

The deduction is as follows:
 Einstein photon energy

- $E = h\nu = h c/\lambda$

Gas microparticles translation energy

- $E = 3/2 k_B T$

When these two energies are equalized, it follows

$$hc/\lambda = 3/2 k_B T \quad and \quad \lambda_T = k_S . 1/ T \text{ (m) is obtained}$$

and

$$k_S = 2/3 (hc/ k_{B.}) = \frac{2. 6,62617.10^{-34} .2,99792458.10^8}{3 . 1, 38066 . 10^{-23}} = 9,592 .10^{-3} \text{ m °k}$$

$$\underline{\lambda_T = 9,592 . 10^{-3} . 1/T} \text{ (m)}$$

Where:

k_S: El-Bahri Sakri constant
k_B: Boltzmann constant
h : Planck constant
C : Light speed

In this relationship (λ, T) it is seen on one hand that the wavelength is independent of gas micro particles mass and it is secondly inversely proportional to temperature. So for sun, the emissivity peak is at a wavelength corresponding to 0.5 microns [1-2] and emission is between 0.1 and 7 microns.

According to the law established in this work by the author, the temperature of sun surface is deduced to be: 19200 °K instead of 5800 °K calculated by Wien's law. This is evidenced by shorter wavelengths given in literature [3] because the temperature is inversely proportional to the wavelength and as wavelength λ is small as the temperature is high and vice-versa.

It is known that sun emits shorter and longer wavelengths. The colors are fairly well represented and our eye sees sun as a white star. Similarly, the temperature surface of the earth is 288 ° K which allows to deduce that most of the terrestrial radiation is in the infrared domain (λ = 30 microns). This relationship joins between temperature and wavelength and creates a link between temperature and color. This allows determining the temperature related to the object color (Table 1).

Table 1

Objet (black body)	Temperature °K	λ_T calculated	Domain spectral
Star type O	50.000	0,180 μm	UV
Sun	19200	0,50 μm	Visible
Earth	300	30 μm	IR
H_2 molecular cloud	20	480 μm	Submm
Cosmic background	3	3 mm	mm

This relationship could be interpreted schematically as follows:

If : $\lambda \to 0$ when $T \to \infty$

       $\lambda \to \infty$ when $T \to 0$

Table 2

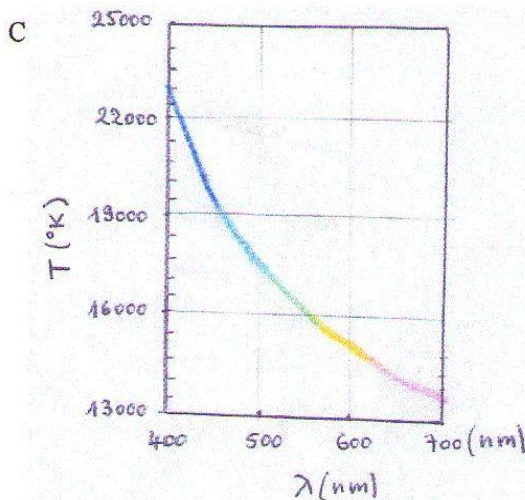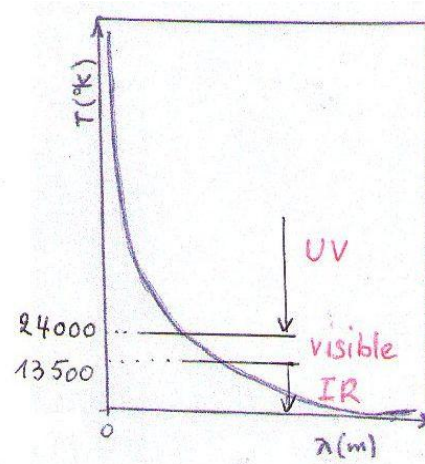| Temprature °k	10	20	100	300	1000	1000000
$\lambda_T\ 10^4$ (m)	9,592	4,80	0,9592	0,317	0,09592	0,00009592

Figure:1: Link between temperature and color.

Figure:2: Variation of temperature vs. wavelength

This relationship (λ, T) has a wide application in astronomy and especially in high temperatures of the heat transfert and the author proposes to simplify the writing of temperature a scale multiple of Kelvin such that:

    1 Kk = 10^3 °k (1 kilo kelvin)
    1 Mk = 10^6 °k (1 Mega kelvin)
    1 Gk = 10^9 °k (1 Gega Kelvin)

A complementary study in quantum chemistry [4-5] shows that there is an abnormal tip in the expression of the radial part R_{nl} (r). This tip shows that the radial part is sometimes positive $R_{nl} > 0$ and sometimes negative R_{nl} (r) < 0. This dual trend value of the radial part results in the polynomials of Laguerre associated $L^{2\ell+1}_{n+\ell}(r)$. In case of ($2\ell + 1 = n + \ell = p$) with p is an odd positive number. We could write:

$$L^{2\ell+1}_{n+\ell}(2zr/na_0) = L^{2\ell+1}_{n+\ell}(r) = L^p_p\ (r)$$

$$L^p_p(r) = \frac{d^p L_P(r)}{dr^p} = \frac{d^p}{dr^p}[\ e^r\ \frac{d^p}{dr^p}\{\ e^{-r}.\ r^p\}]$$

If the calculations are performed correctly, it should be found that:

$$L^p_p(r) = \frac{(-1)^p \cdot p!}{(p-p)!} = (-1)^p \cdot p! \tag{1}$$

Indeed these:

For the atomic orbital 1S ; ($n = 1$, $\ell = 0$)
$$L^{2\ell+1}_{n+\ell}(r) = L^1_1(r) = (-1)^1 \cdot 1! = -1,\text{ which gives a negative radial part which is not logical.}$$

For the atomic orbital 2p ; ($n = 2$, $\ell = 1$)
$$L^{2\ell+1}_{n+\ell}(r) = L^3_3(r) = (-1)^3 \cdot 3! = -6,\text{ same remark as before.}$$

For the atomic orbital 3d ; ($n = 3$, $\ell = 2$)
$$L^{2\ell+1}_{n+\ell}(r) = L^5_5(r) = (-1)^5 \cdot 5! = -120,\text{ always the same remark}$$

Calculations of the radial part from the equation (1) are in conformity with those of the theoretical radial orbitals.
Indeed, for the 1S orbital of a hydrogenoid atom we have:
$$R_{10}(r)_{calcul} = (-1)^1 (2z/a_0)^{3/2} (1/2)^{1/2} \cdot (zr/na_0)^0 \cdot e^{-zr/a0} \cdot L^1_1(2zr/na_0)$$
$$= (-1)^1 (2\sqrt{2})(1/2)^{1/2} (z/a_0)^{3/2} e^{-zr/a_0} (-1)^1$$
$$= 2(z/a_0)^{3/2} \cdot e^{-zr/a0} \quad \text{and the theoretical } R_{10}(r)_{theor} = 2(z/a_0)^{3/2} \cdot e^{-zr/a0}$$
Also for the orbital 2p, we have : .
$$R_{21}(r)_{calcul} = (-1)^3 (2z/2a_0)^{3/2}\{1.0!/4(3!)^3\}^{1/2} \cdot (2zr/2a_0)^1 \cdot e^{-zr/2a0} L^3_3(2zr/na_0)$$
$$= (-1)^3 [1.1/4.(3!)^3]^{1/2} (z/a_0)^{5/2} r\ e^{-zr/2a_0} (-1)^3 \cdot 3!$$
$$= [1.1/4.(3!)] (z/a_0)^{5/2} r\ e^{-zr/a_0}$$
$$= (z/a_0)^{5/2} \cdot r \cdot (1/2\sqrt{6})\ e^{-zr/2a_0} = \frac{1}{2(6)^{1/2}} \cdot (z/a_0)^{5/2} r\ e^{-zr/2a_0} \quad \text{And the theoretical:}$$

$$R_{21}(r)_{theor} = \frac{1}{2(6)^{1/2}} (z/a_0)^{5/2} r\ e^{-zr/2a_0}$$

Similarly we have for the atomic orbital 3d :
$$R_{32}(r)_{calcul} = (-1)^5 (2z/3a_0)^{3/2}(1.0!/6.(5!)^3)^{1/2}(2zr/3a_0)^2 e^{-zr/3a_0} L^5_5(2zr/3a_0)$$
$$= (-1)^5 (2z/3a_0)^{3/2}(1.1/6.(5!)^3)^{1/2}(2zr/3a_0)^2 e^{-zr/3a_0} \cdot (-1)^5 \cdot 5!$$
$$= (2/3)^{7/2} (1.1/6.(5!))^{1/2} (z/a_0)^{7/2} r^2 e^{-zr/3a_0}$$
$$= (2/3)^3 (2/3)^{1/2} (1.1/6.(5!))^{1/2} (z/a_0)^{7/2} r^2 e^{-zr/3a_0}$$
$$= (8/27)(2/6.3.120)^{1/2} (z/a_0)^{7/2} r^2 e^{-zr/3a_0}$$
$$= (8/27)(1/9.4.30)^{1/2} (z/a_0)^{7/2} r^2 e^{-zr/3a_0}\text{ after rearrangement and simplification we coud have:}$$
$$= 4 \cdot (z/a_0)^{7/2} r^2 e^{-zr/3a_0} / 81(30)^{1/2}$$
and the theoretical is given by :
$$R_{32}(r)_{théor} = 4 \cdot (z/a_0)^{7/2} r^2 e^{-zr/3a_0} / 81(30)^{1/2}$$

It is clear that the radial parts $R_{nl}(r)$ calculated and $R_{nl}(r)$ given by the theoretical literature are in full compliance. To remedy this error, the author proposes to rewrite the radial part $R_{nl}(r)$ for different above atomic orbitals and others as follows:

$$R_{nl}(r) = \begin{cases} (-1)^{2\ell+1}\left\{\dfrac{(2z/na_0)^3 (n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2} \cdot (2zr/na_0)^\ell \cdot e^{-zr/na} \cdot L^{2\ell+1}_{n+\ell}(2zr/na_0), & \text{for : } n+\ell = 2\ell+1 = p \\[3mm] \left\{\dfrac{(2z/na_0)^3 (n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2} \cdot (2zr/na_0)^\ell \cdot e^{-zr/na} L^{2\ell+1}_{n+\ell}(2zr/na_0), & \text{for : } n+\ell \neq 2\ell+1 \end{cases}$$

with : a_0 – Bohr atom

It is known that:

$$L^n_p(r) = \frac{(-1)^n\, p!}{(p-n)!}\, e^r\, r^{-n}\, \frac{d^{p-n}}{dr^{p-n}}(e^{-r} r^p) \tag{2}$$

To complete this theoretical research on these hydrogen atom orbitals, we calculated at the end the probability Pn (r) of the electron presence in these orbitals using new mathematical formulas proposed by the author, in fact:

- For the hydrogen atom 1S orbital, the probability is maximum and corresponds to the zero derivative of the function $P_{nl}(r)$ that means $dP_{10}/dr = 0$ and $dP_{10}/dr = \frac{4\pi}{a^3} \cdot 2r\, e^{-2r/a_0}(1 - r/a_0) = 0$. So r =0 and $r = a_0$

The probability of the electron presence according to the equation proposed by the author (Equation 4) is :

$$\int P_n(r)\, e^{\alpha r} = Q_n(r)\, e^{\alpha r} \tag{3}$$

Where : $P_n(r)$ is a polynomial of degree « n » and $Q_n(r)$ is a polynomial of degree « n » with undetermined coefficients. This integral is solved by parts integration (n times). The author proposes to solve it as a simple and effective series as follows:

Where : $$\int_0^{a_0} P_\alpha(r)\, e^{-r/a_0}\, dr = -e^{-r/a_0}\left[\sum_{n=0}^{\alpha} \frac{1}{n!}\, \alpha!\, a^{(\alpha+1-n)}(r)^n\right]\Big|_0^{a_0} \tag{4}$$

With: α – the highest power in "r" the polynome $P_\alpha(r)$

So we have: $P_{10}(r) = \int [\Psi(r, \theta, \varphi)]^2\, r^2 \sin\theta\, dr d\theta d\varphi = [R_{nl}(r)]^2\, r^2\, dr \int_0^{\pi}\int_0^{2\pi} [\,|\,Y_{nl}(\theta, \varphi)\,|\,]^2 \sin\theta\, d\theta d\varphi$

$$= 4\pi r^2\, [R_{nl}(r)]^2$$

–Total 1S wave function of the hydrogen atom

$$\Psi_{100}(r, \theta, \varphi) = \frac{1}{(\pi)^{1/2}}(1/a_0)^{3/2}\, e^{-r/a_0}$$

Using (eq.3) : $$\int r^2\, e^{-2r/a_0}\, dr = (Ar^2 + Br + C)\, e^{-2r/a_0}$$

$$r^2\, e^{-2r/a_0} = (2Ar + B)\, e^{-2r/a_0} - 2/a_0(Ar^2 + Br + C)\, e^{-2r/a_0}$$
$$-2A/a_0 = 1 \longrightarrow A = -a_0/2$$
$$2A - 2B/a_0 = 0 \longrightarrow B = -a_0^2/2$$
$$B - 2C/a_0 = 0 \longrightarrow C = -a_0^3/4$$

Where $$\int_0^{a_0} r^2\, e^{-2r/a_0}\, dr = \frac{-1}{4}(2a_0r^2 + 2a_0^2 r + a_0^3)\, e^{-2r/a_0}\,\Big|_0^{a_0} = \frac{a_0^3}{4}([-5]e^{-2} + 1)$$

We have finally

$$P_{10}(r) = \int_0^{a_0} 4\pi r^2 \psi^2_{100}\, dr = \frac{4\pi a_0^3}{4\pi a_0^3}([-5]e^{-2} + 1) = 1 - 5/e^2 = 0,323 \text{ where } 32,3\%$$

The probability of electron finding in 1S orbital of the hydrogen atom is $P_{10}(r) = 32.3\%$ which is quite similar to the theory.

- Similarly for 2p atomic orbital of the hydrogen atom we have:

$$P_{21}(r) = \int [\Psi(r, \theta, \varphi)]^2\, r^2 \sin\theta\, dr d\theta d\varphi = \frac{1}{16 \cdot 2\pi}(1/a_0)^5 \int r^4\, e^{-2r/2a_0}\, dr \int_0^{\pi}\cos^2\theta \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{1}{16 \cdot 2\pi \cdot 3}(1/a_0)^5 \cdot 2\pi \cdot 2 \int r^4\, e^{-r/a_0}\, dr$$

$\Psi_{210}(r, \theta, \varphi) = \frac{1}{4(2\pi)^{1/2}}(1/a_0)^{5/2}\, r\, e^{-r/2a_0}\cos\theta$ is 2p orbital total wave function of of the hydrogen atom

the maximum probability corresponds to the derivative zero $dP_{21}/dr = 0$ that means, the maximum probability is between r_1 and r_2 that they might be determined

$$dP_{21}/dr = \frac{1}{24} (1/a_0)^5 \; r^3 e^{-r/a_0} (4 - r/a_0) = 0 \;\;, \text{where } r_1 = 0 \;\; \text{and} \;\; r_2 = 4a_0$$

According to equation (3) or (4) we have:

$$\int_0^{4a_0} r^4 \; e^{-r/a_0} dr = (Ar^4 + Br^3 + Cr^2 + Dr + E) \; e^{-r/a_0}$$

$$r^4 \; e^{-r/a_0} = (4Ar^3 + 3Br^2 + 2Cr + D) \; e^{-r/a_0} - 1/a_0 (Ar^4 + Br^3 + Cr^2 + Dr + E) \; e^{-r/a_0}$$

$$-A/a_0 = 1 \;\longrightarrow\; A = -a_0$$
$$4A - B/a_0 = 0 \;\longrightarrow\; B = -4a_0^2$$
$$3B - C/a_0 = 0 \;\longrightarrow\; C = -12a_0^3$$
$$2C - D/a_0 = 0 \;\longrightarrow\; D = -24a_0^4$$
$$D - E/a_0 = 0 \;\longrightarrow\; E = -24a_0^5$$

It finally comes that the probability is equal to:

$$P_{21}(r) = \frac{1}{24 \cdot a_0^5} \cdot \left[-e^{-r/a_0} (a_0 r^4 + 4a_0^2 r^3 + 12a_0^3 r^2 + 24a_0^4 r + 24a_0^5) \right] \Big|_0^{4a_0} = 1 - \frac{824}{24 \, e^4} = 0{,}3709 \;\; \text{or} \;\; 37{,}09\%$$

.

This found value confirms the theory. Also, for the 3d orbital of the hydrogen atom we have:

$$P_{32}(r) = \int [\Psi(r,\theta,\varphi)]^2 \, r^2 \sin\theta \, dr d\theta d\varphi = \frac{1}{(81)^2 \, 6\pi} (1/a_0)^7 \int r^6 e^{-2r/3a_0} dr \int_0^{\pi} (3\cos^2\theta - 1)^2 \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

Where: $\Psi_{320}(r,\theta,\varphi) = \dfrac{1}{81(6\pi)^{1/2}} (1/a_0)^{7/2} \; r^2 e^{-r/3a_0} (3\cos^2\theta - 1)$ is 3d total wave function of the hydrogen atom

We have: $\displaystyle\int_0^{\pi} (3\cos^2\theta - 1)^2 \sin\theta \, d\theta = -[9/5 \cos^5\theta - 2\cos^3\theta + \cos\theta] \Big|_0^{\pi} = 18/5 - 2 = 8/5$

The probability is maximum corresponding to the derivative zero $dP_{nl}/dr = 0$, between r_1 and r_2.

Where: $dP_{nl}/dr = \dfrac{1}{(81)^2 \, 6\pi} (1/a_0)^7 \, \dfrac{8 \cdot 2\pi}{5} \left(r^6 e^{-2r/3a_0} \right)' = K \, r^5 \, e^{-2r/3a_0} (6 - 2r/3a_0) = 0$, so : $r_1 = 0$ and $r_2 = 9a_0$

Similarly and using equation (3) we have: $\int r^6 e^{-2r/3a_0} dr = (Ar^6 + Br^5 + Cr^4 + Dr^3 + Er^2 + Fr + G) \, e^{-2r/3a_0}$
$r^6 e^{-2r/3a_0} = (6Ar^5 + 5Br^4 + 4Cr^3 + 3Dr^2 + 2Er + F) \, e^{-2r/3a_0} - 2/3a_0 (Ar^6 + Br^5 + Cr^4 + Dr^3 + Er^2 + Fr + G) \, e^{-2r/3a_0}$

$$-2A/3a_0 = 1 \;\longrightarrow\; A = -3a_0/2$$
$$6A - 2B/3a = 0 \;\longrightarrow\; B = -27a_0^2/2$$
$$5B - 2C/3a_0 = 0 \;\longrightarrow\; C = -405a_0^3/4$$
$$4C - 2D/3a_0 = 0 \;\longrightarrow\; D = -1215a_0^4/2$$
$$3D - 2E/3a_0 = 0 \;\longrightarrow\; E = -10935a_0^5/4$$
$$2E - 2F/3a_0 = 0 \;\longrightarrow\; F = -32805a_0^6/4$$
$$F - 2G/3a_0 = 0 \;\longrightarrow\; G = -98415a_0^7/8$$

So we have:

$$\int_0^{9a_0} r^6 e^{-2r/3a_0} dr = -\frac{1}{8} e^{-2r/3a_0} (12a_0 r^6 + 108a_0^2 r^5 + 810a_0^3 r^4 + 4860a_0^4 r^3 + 21870a_0^5 r^2 + 65610a_0^6 r + 98415a_0^7) \Big|_0^{9a_0}$$

The probability of the electron presence in 3d orbital of the hydrogen atom is:

$$P_{32}(r) = \frac{1}{(81)^2 \, 6\pi} (1/a_0)^7 \, \frac{8 \cdot 2\pi}{5} \, \frac{a_0^7}{8} \left[98415 - \frac{24072309}{e^6} \right] = 0{,}3933 \;\; \text{where} \;\; 39{,}33\%$$

As before the value of the probability calculated for the 3d orbital of the hydrogen atom is exactly the same as that given in the literature.

Further, it is known that the probability of an electronic transition (the oscillator strength) of the state $\Psi_i = \Psi_{nlm}$(r, θ , φ) to $\Psi_k = \Psi_{n'l'm'}$ (r , θ , φ) state depends straightly of the transition energy and quadratically to the value of the transition dipole moment D_{ki} (Mulliken formula and Rieke):

$$f_{ki} = K\Delta E_{ki}D_{ki}^2$$

where K is the proportionality coefficient which depends on the chosen system of units and is defined as

$$D_{ki} = (\Psi_k \left| \sum_\mu r_\mu \right| \Psi_i)$$

with r_μ is the radius vector of the μth electron of the atom. Transitions, for which $D_{ki} = 0$ are called forbidden in the dipole approximation. Lines corresponding to these transitions are absent in the observed spectrum.

With regard to the emission spectrum of hydrogen atom (Z = 1), the author proposes new formulas similar to that of Mulliken and Rieké that will address the spectrum from the total wave of the orbit Ψ_{ki}(r,θ,φ) and uncertainties read on the frequency and wavelength Δν and Δλ.

Indeed, these formulas are applicable for all hydrogenoid atoms, their form is:

$$\frac{\Delta\lambda}{\lambda^2} = \frac{\Delta\nu}{C} = \frac{D_{ki}|e|\,\varepsilon}{hC} \tag{5}$$

We divide and multiply the equation (5) by the frequency ν, we have:

$$\frac{\Delta\lambda}{\lambda^3} = \frac{D_{ki}|e|\varepsilon\,\nu}{h\,C\,\nu} = \frac{D_{ki}|e|\varepsilon R_H Z^2}{h\,C}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \tag{6}$$

With: D_{ki} (a_0) = D'. a_0/z (m) -- proportionality factor that takes into account the initial state $\Psi_k(r,\theta,\varphi)$ and the final state $\Psi_i(r,\theta,\varphi)$, or : D' – number addimensional
ε - electric field strength , V/m

To justify the validity of these formulas, we compare the spectrum found by these formulas to Lyman, Balmer, Pashen ...etc given in the literature for the hydrogen atom and all hydrogenoid atoms

For the first line of Lyman, we have the transition $\Psi_{210} \rightarrow \Psi_{100}$ or ($n_2=2 \rightarrow n_1=1$), To do this, first calculate the D_{ki} factor as: $D_{ki} = \Psi_k|z|\Psi_i$, along the axis « oz » with z = rcosθ, under an electric field strength ε, (Stark effect on dégénérescence levels).

With
$$\Psi_{210} = \frac{1}{(8\pi)^{1/2}}\,.1\,(Z/a_0)^{5/2}\,r\,e^{-Zr/2a_0}\cos\theta$$

$$\Psi_{100} = \frac{1}{(\pi)^{1/2}}(Z/a_0)^{3/2}\,e^{-zr/a_0}$$

Then: $D_{ki} = \int_0^\infty \int_0^\pi \int_0^{2\pi} R^*_{nl}(r)[Y^*_l(\theta,\varphi)]^*\, rcos\theta R_{nl}(r)Y^m_l(\theta,\varphi)r^2 sin\theta\, dr\, d\theta\, d\varphi$

$$D_{ki} = \frac{1}{2(8\pi^2)^{1/2}}.1\,(z/a_0)^4 \int_0^\infty r^4 e^{-r(1z/a_0 + 1z/2a_0)}dr\int_0^\pi cos^2\theta sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{1(z/a_0)^4}{(8)^{1/2}} \int_0^\infty r^4 e^{-r(3/2a_0)}dr\int_0^\pi cos^2\theta sin\theta d\theta$$

With:
$$\int_0^\infty r^4 e^{-r(3z/2a_0)}dr = \frac{4!}{(3z/2a_0)^5} \qquad\text{and}\qquad \int_0^\pi cos^2\theta sin\theta d\theta = 2/3$$

Where : $D_{ki} = \dfrac{1.\,24.64}{(8)^{1/2}\,720}. (z/a_0)^4(a_0/z)^5 = 0{,}745\ a_0/z$ (m)

And : $\Delta\nu = \dfrac{D_{ki}|e|\varepsilon}{h} = \dfrac{0{,}745|e|\varepsilon a_0}{h} = \dfrac{0{,}745\ 1{,}60\ 10^{-19}\ 10^6\ 0{,}53\ 10^{-10}}{6{,}62\ 10^{-34}} = 9{,}54\ 10^9\ s^{-1}$

Then : $\dfrac{\Delta\lambda}{\lambda^2} = \dfrac{\Delta v}{C} = \dfrac{D_{ki}|e|\varepsilon}{hC} = \dfrac{9,54\ 10^9}{3\ .\ 10^8} = 31,8\ $ m^{-1}

$$\dfrac{\Delta\lambda}{\lambda^3} = \dfrac{D_{ki}|e|\ \varepsilon\ R_H Z^2}{hC}\left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right) = \dfrac{0,754.\ 1,60.10^{-19}.10^6\ .0,53.\ 10^{-10}.\ 1,1.\ 10^7(1/1\ -1/4)}{6,62\ .10^{-34}\ 3\ 10^8} = 2,624\ 10^8\ \text{m}^{-2}$$

We divide $(\Delta\lambda/\lambda^2)/(\Delta\lambda/\lambda^3)$ which gives us the calculated length $\lambda_{\text{calculated}} = \dfrac{31,8}{2,624\ 10^8} = 1212$ Å

the wavelength of the first line of Lyman given by the literature is: $\lambda_{\text{theoretical}} = 1212,12$Å. We can see that there is an excellent match between the theory and the calculation. Generally, all the transitions are not allowed because $D_{ki}=0$, such as $\Psi_{200}\rightarrow\Psi_{100}$, $\Psi_{320}\rightarrow\Psi_{100}$, $\Psi_{420}\rightarrow\Psi_{210}$ …etc.
while: $\Psi_{320}\rightarrow\Psi_{210}$ is the first line of Balmer and $\Psi_{430}\rightarrow\Psi_{320}$ is the first line of Pashen, so we can get all the lines of each series of hydrogenoid atoms by these formulas.

So the first line of Balmer $\Psi_{320}\rightarrow\Psi_{210}$ or between ($n_2 = 3\rightarrow n_2= 2$) is equal to:

$$\Psi_{320} = \dfrac{1}{81(6\pi)^{1/2}}(z/a_0)^{7/2}\ r^2\ e^{-zr/3a_0}\ (3\cos^2\theta -1)$$
$$\Psi_{210} = \dfrac{1}{4(2\pi)^{1/2}}(z/a_0)^{5/2}\ r\ e^{-zr/2a_0}\ \cos\theta$$

Then:

$$D_{ki} = \dfrac{1}{81.4\ .(12)^{1/2}\pi}\ (z/a_0)^6\ \int_0^\infty r^6\ e^{-r(5z/6a_0)}\ dr \int_0^\pi (\cos^2\theta - 1)\ \cos^2\theta\sin\theta\ d\theta \int_0^{2\pi} d\varphi$$

With: $\displaystyle\int_0^\infty r^6\ e^{-r(5z/6a_0)}dr = \dfrac{6\ !}{(5z/6a_0)^7}$

And: $\displaystyle\int_0^\pi (3\cos^4\theta - \cos^2\theta)\sin\theta d\theta = 8/15$

So: $D_{ki} = \dfrac{1.1.8.720.279936\ (\ a_0/z)}{324\ (3)^{1/2}.15.78125} = 2,451\ a_0/z$ (m)

And: $\dfrac{\Delta\lambda}{\lambda^2} = \dfrac{\Delta v}{C} = \dfrac{D_{ki}|e|\varepsilon}{hC} = \dfrac{2,451\ .1,60.\ 10^{-19}.10^6.\ 0,53.10^{-10}}{6,62\ .10^{-34}\ 3\ .10^8} = 104,65\ $ m^{-1}

$$\dfrac{\Delta\lambda}{\lambda^3} = \dfrac{D_{ki}|e|\varepsilon\ R_H Z^2}{hC}\left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right) = \dfrac{2,451\ .1,60.10^{-19}.10^6.0,53.10^{-10}.1,1\ .10^7\ .(1/4 - 1/9)}{6,62\ .\ 10^{-34}\ .\ 3\ .10^8} = 15,989\ .10^7\ \text{m}^{-2}$$

So the wavelength of the first Balmer line for hydrogen (Z = 1) is: $(\Delta\lambda/\lambda^2)/(\Delta\lambda/\lambda^3)$
is equal to: $\lambda_{\text{calculated}} = \dfrac{104,65}{15,989\ .10^7} = 6545,53$ Å , the theoretical value of this wavelength for the same atom is
$\lambda_{\text{theoretical}} =6545,45$Å.We can say that the calculated value fits well with the theoretical one.

Also for the first line of Pashen between the total wave functions $\Psi_{430}\rightarrow\Psi_{320}$ or between ($n_2=4\rightarrow n_1=3$).
Namely:

$$\Psi_{430} = \dfrac{1}{384} (7/6\pi140)^{1/2}(z/a_0)^{9/2} r^3\ e^{-zr/4a_0}(5\cos^3\theta -3\cos\theta)$$
$$\Psi_{320} = \dfrac{1}{81\ (6\pi)^{1/2}} (z/a_0)^{7/2}\ r^2\ e^{-zr/3a_0}(3\cos^2\theta -1)$$

the D_{ki} factor is first calculated

$$D_{ki} = \dfrac{1}{384.81.6\pi(20)^{1/2}}\ (z/a_0)^8\ \int_0^\infty r^8\ e^{-r(7z/12a_0)}dr \int_0^\pi (5\cos^3\theta - 3\cos\theta)(3\cos^2\theta -1)\cos\theta\sin\theta d\theta\int_0^{2\pi} d\varphi$$

$\infty \qquad\qquad\qquad \pi$

$$= \frac{1}{384.81.3.4,472} \; (z/a_0)^8 \quad \int_0^\infty r^8 \, e^{-r(7z/12a_0)} dr \int_0 (15\cos^5\theta - 14\cos^3\theta + 3\cos\theta)\cos\theta\sin\theta d\theta$$

With :
$$\int_0^\infty r^8 e^{-r(7z/12a_0)} \, dr = \frac{8\,!}{(7z/12a_0)^9}$$

$$\int_0^\pi (15\cos^6\theta - 14\cos^4\theta + 3\cos^2\theta)\sin\theta d\theta = 24/35$$

Then: $D_{ki} = \dfrac{5155483,221\,.24\;(z/a_0)^8 (a_0/z)^9}{384.81.3.4,472.35} = 8,471\; a_0/z$ (m)
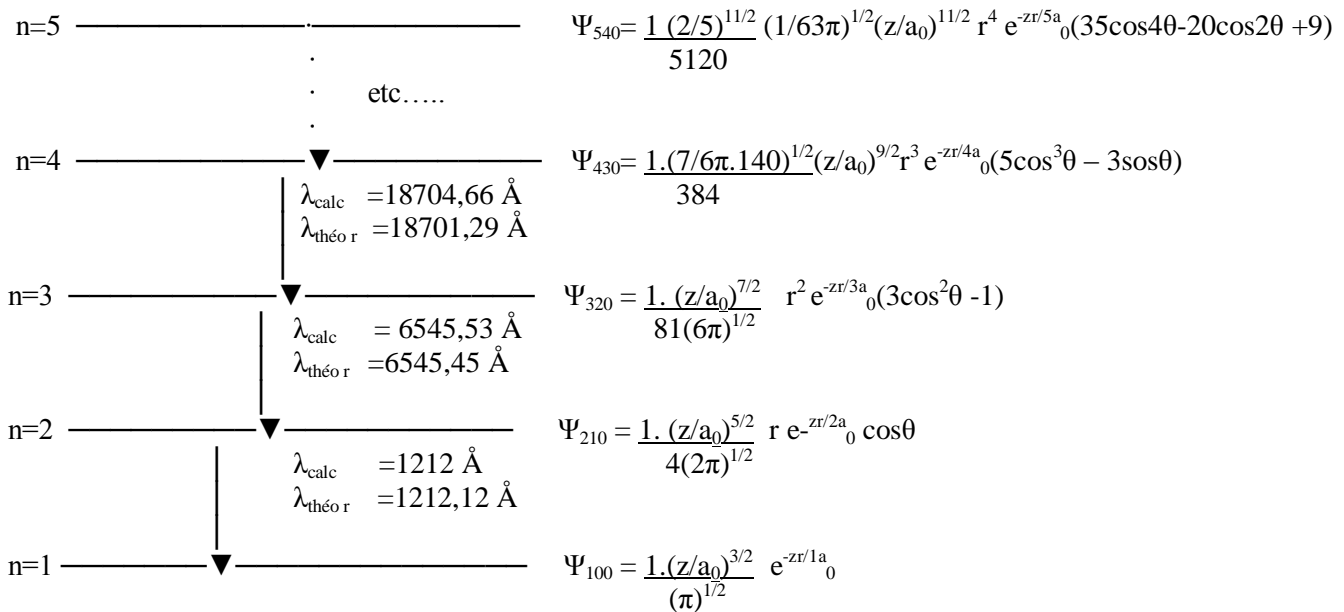
And :

$$\frac{\Delta\lambda}{\lambda^2} = \frac{\Delta v}{C} = \frac{D_{ki}|e|\varepsilon}{hC} = \frac{8,471\,.1,60.10^{-19}.10^6.0,53\,.10^{-10}}{6,62\,.10^{-34}\,.\,3\,.10^8} = 361 \;\; m^{-1}$$

$$\frac{\Delta\lambda}{\lambda^3} = \frac{D_{ki}|e|\varepsilon\,R_H Z^2}{hC}\left(\frac{1}{n_1^{\,2}} - \frac{1}{n_2^{\,2}}\right) = \frac{8,471.1,60.10^{-19}.10^6.0,53.10^{-10}.1,1.10^7.(1/9 - 1/16)}{6,62.10^{-34}\,.3.10^8} = 1,93\,.10^8 \; m^{-2}$$

Indeed, the first line of Pashen is equal to: $(\Delta\lambda/\lambda^2)/(\Delta\lambda/\lambda^3) \rightarrow \lambda_{calculated} = \dfrac{0,361\,.10^3}{1,93\,.10^8} = 18704,66$ Å

According to the literature, this line has a wavelength equal to: $\lambda_{theoretical} = 18701,29$Å, in this case it is clear that the theory coincides with the calculations and there is between theory and calculation an excellent match.
In summary, the results found can be represented as follows:

n=5 ——————————·————————— $\Psi_{540} = \dfrac{1\,(2/5)^{11/2}\,(1/63\pi)^{1/2}(z/a_0)^{11/2}\,r^4\,e^{-zr/5a_0}(35\cos4\theta - 20\cos2\theta + 9)}{5120}$

         ·   etc…..

n=4 ——————————▼————————— $\Psi_{430} = \dfrac{1.(7/6\pi.140)^{1/2}(z/a_0)^{9/2} r^3\,e^{-zr/4a_0}(5\cos^3\theta - 3\text{sos}\theta)}{384}$

      $\lambda_{calc} = 18704,66$ Å
      $\lambda_{théo\,r} = 18701,29$ Å

n=3 ——————————▼————————— $\Psi_{320} = \dfrac{1.\,(z/a_0)^{7/2}}{81(6\pi)^{1/2}}\; r^2\,e^{-zr/3a_0}(3\cos^2\theta - 1)$

      $\lambda_{calc} = 6545,53$ Å
      $\lambda_{théo\,r} = 6545,45$ Å

n=2 ——————————▼————————— $\Psi_{210} = \dfrac{1.\,(z/a_0)^{5/2}}{4(2\pi)^{1/2}}\; r\,e^{-zr/2a_0}\cos\theta$

      $\lambda_{calc} = 1212$ Å
      $\lambda_{théo\,r} = 1212,12$ Å

n=1 ——————————▼————————— $\Psi_{100} = \dfrac{1.(z/a_0)^{3/2}}{(\pi)^{1/2}}\; e^{-zr/1a_0}$

Conclusion

The theoretical relationship $(\lambda,\, T)$ seems to be more close to reality in two ways:
- The temperature on the surface of sun found by the author is much higher than that found by Wien's law which is quite accurate.

- The short wavelengths emitted by the sun justify its high surface temperature. The radial part $R_{nl}(r)$ can be now calculated with great precision, always it takes a positive value which makes sense regardless of the sign of the associated Laguerre polynomial. The radial parts calculated according to new mathematical formula (éq ; 1- 4) and those given in the literature are in absolut conformity.

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This relationship can be applied quite exceptional in the field of astronomy and astrophysics; it lets known the temperature of the stars and objects (black body) from far, by simply reading the wavelength. The same notice that we can make out about the probability of electron's within these otbitals.

Regarding the spectrum of atomic hydrogen, the latter has been described with great precision from the calculations (eq . 5-6) of the wavelength of the first spectral line of each series, compared with that given in the literature. This comparison between theory and calculations is found to be excellent. Finally, we can say firmly that all was said about the hydrogen atom, may actually be said about all other hydrogenoid atoms.

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