ORBITAL GRAVITATION AND ORBITAL HAUSDORFF STABILITY OF LOTKA-VOLTERRA MODEL

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Abstract

The main object of investigation in the present paper is the classical Lotka-Volterra mathematical model. There are introduced orbital gravitation and orbital Hausdorff stability of the trajectories of this model. Under natural assumptions, it is showed that Lotka-Volterra model possesses these properties.

Key words: model of Lotka-Volterra, trajectory, orbital gravitation, orbital Hausdorff stability.

1. Introduction

The Lotka-Volterra mathematical model describes quite accurately the evolution dynamics of predator-prey interactions of an isolated (without external influences) biosystem. The classical initial problem of this model has a form:

(1) $$\frac{dm}{dt}=\dot{m}=F_m(m,M)=m(r_1-q_1M),$$

(2) $$\frac{dM}{dt}=\dot{M}=F_M(m,M)=-M(r_2-q_2m),$$

(3) $$m(0)=m_0;\quad M(0)=M_0,$$

Where:

- $m=m(t)>0$ and $M=M(t)>0$ are the quantities of biomasses of the prey and the predator respectively at the moment $t\geq 0$;
- The constants $r_1>0$ and $r_2>0$ are specific coefficients of the relative growth of the first species (prey) and the second species (predator), respectively;
- The constants $q_1>0$ and $q_2>0$ are the coefficients reflecting interspecies competition for the prey and the predator, respectively;
- The constants $m_0>0$ and $M_0>0$ are the quantities of biomasses of both species at the initial moment $t=0$.

It is known that the system (1), (2) possesses:

- Unstable stationary point $(0,0)$, (the origin is a saddle point);
- Stable stationary point $(m_{00},M_{00})=\left(r_2/q_2,\, r_1/q_1\right)$;
- A first integral of the following form

$$U(m,M)=q_1M+q_2m-r_1\ln M-r_2\ln m+r_1\left(\ln r_1/q_1-1\right)+r_2\left(\ln r_2/q_2-1\right)$$
$$=W(m,M)-W(m_{00},M_{00}),$$

where

$$W(m,M)=q_1M+q_2m-r_1\ln M-r_2\ln m;$$

- For any point $(m,M)\in\mathbb{R}^+\times\mathbb{R}^+$, $(m,M)\neq(m_{00},M_{00})$ the inequality $U(m,M)>0$ is valid. It is fulfilled $U(m_{00},M_{00})=0$;
- For any constant $c\geq 0$ the implicitly given curve

$$\gamma_c=\{(m,M):U(m,M)=c\}$$

is a trajectory of the system (1), (2) with a properly chosen initial condition (it is sufficient to assume that $U(m_0,M_0)=c$);
- For any constant $c>0$ the set

$$D_c=\{(m,M):U(m,M)<c\}$$

is a connected domain, located in $\mathbb{R}^+\times\mathbb{R}^+$, with a contour $\partial D_c=\gamma_c$;
- For any constant $c>0$ it is satisfied $(m_{00},M_{00})\in D_c$;
- If $0<c_1<c_2$, then $\gamma_{c_1}\in D_{c_2}$.

Different aspects of the population dynamics are studied in [1] ÷ [27].

2. Statement of the problem and preliminary remarks

If the points $a(a_1,a_2,...,a_n)$, $b(b_1,b_2,...,b_n)\in\mathbb{R}^n$, then their dot product, the Euclidean norm and the Euclidean distance between them are denoted respectively by:

$$\langle a,b\rangle=a_1b_1+a_2b_2+...+a_nb_n,$$

$$\|a\|=\langle a,a\rangle^{1/2}=\sqrt{a_1^{\,2}+a_2^{\,2}+...+a_n^{\,2}},$$

$$\rho_E(a,b)=\sqrt{(a_1-b_1)^2+(a_2-b_2)^2+...+(a_n-b_n)^2}.$$

It is clear that equality $\|a-b\|=\rho_E(a,b)$ is valid. If the non empty sets $A,\ B\subset\mathbb{R}^n$, then the Euclidean and the Hausdorff distances between them are denoted respectively by:

$$\rho_E(A,B)=\inf\{\inf\{\rho_E(a,b),a\in A\},b\in B\},$$

$$\rho_H(A,B)=\max\{\sup\{\inf\{\rho_E(a,b),a\in A\},b\in B\},\ \sup\{\inf\{\rho_E(a,b),b\in B\},a\in A\}\}.$$

The inequality $\rho_E(A,B)\leq\rho_H(A,B)$ is obviously true.

The Euclidean and the Hausdorff distance between the trajectories γ_{c_0} and $\gamma_{c_0^*}$ satisfy the following equalities respectively:

$$\rho_E\left(\gamma_{c_0^*},\gamma_{c_0}\right)=\inf\left\{\inf\left\{\rho_E\left((m^*,M^*),(m,M)\right),\ (m^*,M^*)\in\gamma_{c_0^*}\right\},\ (m,M)\in\gamma_{c_0}\right\},$$

$$\rho_H\left(\gamma_{c_0^*},\gamma_{c_0}\right)=\max\left\{\sup\left\{\inf\left\{\rho_E\left((m^*,M^*),(m,M)\right),\ (m^*,M^*)\in\gamma_{c_0^*}\right\},\ (m,M)\in\gamma_{c_0}\right\},\right.$$

$$\left.\sup\left\{\inf\left\{\rho_E\left((m^*,M^*),(m,M)\right),(m,M)\in\gamma_{c_0}\right\},\ (m^*,M^*)\in\gamma_{c_0^*}\right\}\right\}.$$

Definition 1. *We say that the system (1), (2) is orbital gravitating in the domain* D *with a constant* $\kappa \geq 1$*, if:*

$$\left(\forall c_0^*, c_0 \in \mathbb{R}^+\right):\left(\gamma_{c_0^*},\, \gamma_{c_0} \in D\right) \Rightarrow \rho_H\left(\gamma_{c_0^*},\, \gamma_{c_0}\right) \leq \kappa.\rho_E\left(\gamma_{c_0^*},\, \gamma_{c_0}\right)$$

$$\Leftrightarrow \max\left\{\sup\left\{\inf\left\{\rho_E\left(\left(m^*, M^*\right),\left(m, M\right)\right),\, \left(m^*, M^*\right)\in\gamma_{c_0^*}\right\},\, \left(m, M\right)\in\gamma_{c_0}\right\},\right.$$

$$\left.\sup\left\{\inf\left\{\rho_E\left(\left(m^*, M^*\right),\left(m, M\right)\right),\left(m, M\right)\in\gamma_{c_0}\right\},\; \left(m^*, M^*\right)\in\gamma_{c_0^*}\right\}\right\}$$

$$\leq \kappa.\inf\left\{\inf\left\{\rho_E\left(\left(m^*, M^*\right),\left(m, M\right)\right),\, \left(m^*, M^*\right)\in\gamma_{c_0^*}\right\},\, \left(m, M\right)\in\gamma_{c_0}\right\}.$$

Definition 2. *We say that the solution of problem (1), (2), (3) is orbital Hausdorff stable if:*

$$\left(\forall\varepsilon > 0\right)\left(\forall\left(m_0, M_0\right)\in\mathbb{R}^+\times\mathbb{R}^+\right)\left(\exists\delta = \delta\left(\varepsilon, m_0, M_0\right) > 0\right):$$

$$\left(\forall\left(m_0^*, M_0^*\right)\in\mathbb{R}^+\times\mathbb{R}^+,\ \rho_E\left(\left(m_0^*, M_0^*\right),\left(m_0, M_0\right)\right) < \delta\right) \Rightarrow\ \rho_H\left(\gamma_{c_0^*},\gamma_{c_0}\right) < \varepsilon\,,$$

where $c_0 = U\left(m_0, M_0\right)$ *and* $c_0^* = U\left(m_0^*, M_0^*\right)$.

The following two theorems are auxiliary.

Theorem 1. *Assume that:*

1. *The constants* c_0 *and* c_0^* *satisfy the inequalities* $0 < c_0 < c_0^*$;
2. *The domain* $D = D_{c_0^*} \setminus D_{c_0}$.

Then for every point $\left(m, M\right)\in\gamma_{c_0}$ *there exists a point* $\left(m^*, M^*\right)\in\gamma_{c_0^*}$ *such that the segment*

$$\mu = \left\{\left(m_\mu, M_\mu\right);\ m_\mu = \left(1-\lambda\right)m + \lambda m^*,\ \ M_\mu = \left(1-\lambda\right)M + \lambda M^*,\ 0 \leq \lambda \leq 1\right\} \subset \overline{D}.$$

Proof. Let the point $\left(m, M\right)\in\gamma_{c_0}$. We consider the half-line

$$sl = \left\{\left(m + \lambda'\left(m - m_{00}\right),\ M + \lambda'\left(M - M_{00}\right)\right),\ \ \lambda' \geq 0\right\}.$$

There exists a constant $\lambda^* > 0$ such that

$$\left(m + \lambda^*\left(m - m_{00}\right),\ M + \lambda^*\left(M - M_{00}\right)\right) = \left(m^*, M^*\right)\in\gamma_{c_0^*}.$$

It is true that

$$\mu = \left\{\left(\left(1-\lambda\right)m + \lambda m^*, \left(1-\lambda\right)M + \lambda M^*\right),\ 0 \leq \lambda \leq 1\right\}$$

$$= \left\{\left(m + \lambda'\left(m - m_{00}\right),\ M + \lambda'\left(M - M_{00}\right)\right),\ \ 0 \leq \lambda' \leq \lambda^*\right\},$$

where $\lambda = \lambda'/\lambda^*$. We shall show that $\mu \in \overline{D}$.

One of the following cases is valid:

1. $m \leq m_{00},\ M \leq M_{00}$;
2. $m < m_{00},\ M > M_{00}$;
3. $m > m_{00},\ M < M_{00}$;
4. $m \geq m_{00},\ M \geq M_{00}$,

where $m_{00} = {}^{r_2}\!\big/_{q_2}$ and $M_{00} = {}^{r_1}\!\big/_{q_1}$. We shall consider only case 1. In this case we get:

$$q_2 - {}^{r_2}\!\big/_{m} \le q_2 - {}^{r_2}\!\big/_{m_{00}} = q_2 - {}^{r_2}\!\big/_{r_2/q_2} = 0, \quad q_1 - {}^{r_1}\!\big/_{M} \le q_1 - {}^{r_1}\!\big/_{M_{00}} = q_1 - {}^{r_1}\!\big/_{r_1/q_1} = 0.$$

Let the function $F(\lambda') = U\left(\left(m + \lambda'(m - m_{00}),\, M + \lambda'(M - M_{00})\right)\right)$, $0 \le \lambda' \le \lambda^*$. We have

$$\frac{d}{d\lambda'} F(\lambda') = \frac{d}{d\lambda'} U\left(\left(m + \lambda'(m - m_{00}),\, M + \lambda'(M - M_{00})\right)\right)$$
$$= \frac{d}{d\lambda'} W\left(\left(m + \lambda'(m - m_{00}),\, M + \lambda'(M - M_{00})\right)\right)$$
$$= \left(q_2 - \frac{r_2}{m + \lambda'(m - m_{00})}\right)(m - m_{00}) + \left(q_1 - \frac{r_1}{M + \lambda'(M - M_{00})}\right)(M - M_{00}) \ge 0,$$

because

$$m - m_{00} \le 0 \;, \quad q_2 - \frac{r_2}{m + \lambda'(m - m_{00})} \le q_2 - \frac{r_2}{m} \le 0,$$
$$M - M_{00} \le 0 \;, \quad q_1 - \frac{r_1}{M + \lambda'(M - M_{00})} \le q_1 - \frac{r_1}{M} \le 0.$$

Therefore, $F(0) \le F(\lambda') \le F(\lambda^*)$, i.e.

$$c_0 = U(m, M) = F(0) \le F(\lambda')$$
$$= U\left(\left(m + \lambda'(m - m_{00}),\, M + \lambda'(M - M_{00})\right)\right)$$
$$\le F(\lambda^*) = U(m^*, M^*) = c_0^*, \quad 0 \le \lambda' \le \lambda^*.$$

From last inequalities we conclude that $\left(\left(m + \lambda'(m - m_{00}),\, M + \lambda'(M - M_{00})\right)\right) \in \overline{D},\ 0 \le \lambda' \le \lambda^*$ or $\mu \in \overline{D}$.

The theorem is proved.

Theorem 2. *Assume that:*

1. *The constants* c_0 *and* c_0^* *satisfy the inequalities* $0 < c_0 < c_0^*$;
2. *The connected set* $\gamma \subset D = D_{c_0^*} \setminus D_{c_0}$;
3. *For every points* $(m, M) \in \gamma_{c_0}$ *and* $(m^*, M^*) \in \gamma_{c_0^*}$ *the segment*

$$\mu = \left\{(m_\mu, M_\mu);\ m_\mu = (1-\lambda)m + \lambda m^*,\ M_\mu = (1-\lambda)M + \lambda M^*,\ 0 \le \lambda \le 1\right\}$$

intersects the set γ, *i.e.* $\gamma \cap \mu \ne \varnothing$.

Then for every point $(m, M) \in \gamma_{c_0}$ *there exist the points* $(m^*, M^*) \in \gamma_{c_0^*}$ *and* $(m', M') \in \gamma \cap \mu$ *such that the vector* $(m^* - m,\ M^* - M)$ *is collinear with* $\operatorname{grad} U(m', M')$.

Proof. Let the point $(m, M) \in \gamma_{c_0}$. Similarly of the proof of previous theorem one of the following cases is valid:

1. $m \le m_{00}$, $M \le M_{00}$;
2. $m < m_{00}$, $M > M_{00}$;
3. $m > m_{00}$, $M < M_{00}$;
4. $m \ge m_{00}$, $M \ge M_{00}$.

Case 1. $m \le m_{00}$, $M \le M_{00}$. We denote:

$$\gamma'_{c_0^*} = \gamma_{c_0^*} \cap \left\{ \left(m^*, M^*\right),\ m^* \le m,\ M^* \le M \right\},$$
$$\gamma' = \gamma \cap \left\{ \left(m', M'\right),\ m' \le m,\ M' \le M \right\}.$$

For each point $\left(m', M'\right) \in \gamma'$ there are satisfied:

$$gradU\left(m', M'\right) = \left(\frac{\partial U\left(m', M'\right)}{\partial m}, \frac{\partial U\left(m', M'\right)}{\partial M} \right) = \left(q_2 - {r_2}/{m'},\ q_1 - {r_1}/{M'} \right),$$
$$q_2 - {r_2}/{m'} \le q_2 - {r_2}/{m} \le q_2 - {r_2}/{m_{00}} = q_2 - \frac{r_2}{r_2/q_2} = 0, \qquad q_1 - {r_1}/{M'} \le 0.$$

Case 1.1. Let $m < m_{00}$ and $M < M_{00}$. Then the following strict inequalities are valid:

$$q_2 - {r_2}/{m'} < 0, \qquad q_1 - {r_1}/{M'} < 0.$$

Let μ is the segment with the endpoints $\left(m, M\right) \in \gamma_{c_0}$ and $\left(m^*, M^*\right) \in \gamma_{c_0^*}$. For every point $\left(m^*, M^*, m', M'\right) \in \gamma'_{c_0^*} \times \left(\gamma' \cap \mu\right)$ we consider the function

$$F\left(m^*, M^*, m', M'\right) = \frac{m^* - m}{q_2 - {r_2}/{m'}} - \frac{M^* - M}{q_1 - {r_1}/{M'}},$$

First, let the point $\left(m_1^*,\ M_1^*\right) = \left(m,\ M_1^*\right) \in \gamma'_{c_0^*}$. Then, it is clear that $M_1^* < M$ and therefore

$$F\left(m_1^*, M_1^*, m', M'\right) = \frac{m - m}{q_2 - {r_2}/{m'}} - \frac{M_1^* - M}{q_1 - {r_1}/{M'}} < 0.$$

Second, if the point $\left(m_2^*,\ M_2^*\right) = \left(m_2^*,\ M\right) \in \gamma'_{c_0^*}$, then $m_2^* < m$. We get

$$F\left(m_2^*, M_2^*, m', M'\right) = \frac{m_2^* - m}{q_2 - {r_2}/{m'}} - \frac{M - M}{q_1 - {r_1}/{M'}} > 0.$$

Since the function F is continuous on the connected set $\gamma'_{c_0^*} \times \left(\gamma' \cap \mu\right)$ then there exists a point $\left(m^*,\ M^*, m', M'\right)$ from this set such that

$$F\left(m^*, M^*, m', M'\right) = 0 \iff \frac{m^* - m}{q_2 - {r_2}/{m'}} = \frac{M^* - M}{q_1 - {r_1}/{M'}}.$$

This proves the theorem in case 1.1.

Case 1.2. Let $m = m_{00}$ and $M \le M_{00}$. We assume that $m^* = m$. Then the following equalities are valid:

$$m^* = m = m_{00} = m', \qquad q_2 - r_2/m' = 0.$$

and consequently

$$\left(m^* - m\ ,\ M^* - M\right) = \left(0\ ,\ M^* - M\right),$$

$$gradU\left(m', M'\right) = \left(0\ ,\ q_1 - r_1/M'\right).$$

From the last equalities it follows that the vectors $\left(m^* - m\ , M^* - M\right)$ and $gradU\left(m', M'\right)$ are collinear.

Case 1.3. Let $m \le m_{00}$ and $M = M_{00}$. Then, as the previous case we determine

$$\left(m^* - m\ ,\ M^* - M\right) = \left(m^* - m\ ,\ 0\right),$$

$$gradU\left(m',\ M'\right) = \left(q_2 - r_2/m'\ ,\ 0\right).$$

Therefore, the above vectors are collinear. Under this assumption the theorem is proved.

The remaining three cases: $m < m_{00}$, $M > M_{00}$; $m > m_{00}$, $M < M_{00}$ and $m \ge m_{00}$, $M \ge M_{00}$ are considered similarly so we omit them.

The theorem is proved.

3. Main result

Theorem 3. *Assume that:*

1. *The constants* c_1 *and* c_2 *satisfy the inequalities* $0 < c_1 < c_2$*;*
2. *The domain* $D = D_{c_2} \setminus D_{c_1}$.

Then the system (1), (2) is orbital gravitating in the domain D *with a constant*

$$\kappa = \frac{\sup\left\{\left\|gradU\left(m', M'\right)\right\|, \left(m', M'\right) \in D\right\}}{\inf\left\{\left\|gradU\left(m', M'\right)\right\|, \left(m', M'\right) \in D\right\}}.$$

Proof. Let the trajectories $\gamma_{c_0}, \gamma_{c_0^*} \subset D$. We have $c_1 < c_0 < c_2$, $c_1 < c_0^* < c_2$ and

$$\begin{aligned}\gamma_{c_0} &= \left\{\left(m, M\right) : U\left(m, M\right) = c_0\right\}\\ &= \left\{\left(m, M\right):\ q_1 M + q_2 m - r_1 \ln M - r_2 \ln m + r_1\left(\ln r_1/q_1 - 1\right) + r_2\left(\ln r_2/q_2 - 1\right) = c_0\right\},\end{aligned}$$

$$\gamma_{c_0^*} = \left\{\left(m, M\right):\ q_1 M + q_2 m - r_1 \ln M - r_2 \ln m + r_1\left(\ln r_1/q_1 - 1\right) + r_2\left(\ln r_2/q_2 - 1\right) = c_0^*\right\}.$$

We assume that $c_0 < c_0^*$. The proof of the case $c_0 > c_0^*$ is similar. The case $c_0 = c_0^*$ is trivial.

Let the point $\left(m, M\right) \in \gamma_{c_0}$. Let $\left(m^*, M^*\right)$ be a point such that $\left(m^*, M^*\right) \in \gamma_{c_0^*}$ and the segment μ with the endpoints $\left(m, M\right)$ and $\left(m^*, M^*\right)$ belongs to $\overline{D_{c_0^*} \setminus D_{c_0}}$ (see Theorem 1). We consider a function

$$F\left(\lambda\right) = U\left(m + \lambda\left(m^* - m\right), M + \lambda\left(M^* - M\right)\right), \quad \lambda \in [0,1].$$

We have: $F(0)=U(m,M)=c_0$, $F(1)=U(m^*,M^*)=c_0^*$ and F is continuous differentiable function on the interval $[0,1]$. Then there exists at last one constant $\lambda_0=\lambda_0(m,\ M\ ,\ m^*,\ M^*)$, $0<\lambda_0<1$, such that

(4) $$\begin{aligned}|c_0^*-c_0| &= |F(1)-F(0)| = |F'(\lambda_0)| \\ &= \left|\frac{d}{d\lambda}U\left(m+\lambda_0\left(m^*-m\right),M+\lambda_0\left(M^*-M\right)\right)\right| \\ &= \left|\frac{\partial}{\partial m}U\left(m+\lambda_0\left(m^*-m\right),M+\lambda_0\left(M^*-M\right)\right)\left(m^*-m\right)\right. \\ &\quad \left.+\frac{\partial}{\partial M}U\left(m+\lambda_0\left(m^*-m\right),M+\lambda_0\left(M^*-M\right)\right)\left(M^*-M\right)\right|. \\ &= \left|\left\langle gradU\left(m+\lambda_0\left(m^*-m\right),M+\lambda_0\left(M^*-M\right)\right),\left(m^*-m,\ M^*-M\right)\right\rangle\right|\end{aligned}$$

$$=\left|\left\langle gradU\left(m+\lambda_0\left(m^*-m\right),M+\lambda_0\left(M^*-M\right)\right),\right.\right.$$
$$\left.\left.\left(\frac{m^*-m}{\sqrt{\left(m^*-m\right)^2+\left(M^*-M\right)^2}},\frac{M^*-M}{\sqrt{\left(m^*-m\right)^2+\left(M^*-M\right)^2}}\right)\right\rangle\right|.\sqrt{\left(m^*-m\right)^2+\left(M^*-M\right)^2}.$$

We denote

$$\begin{aligned}\gamma=\Big\{&\left(m',M'\right)=\left(m+\lambda_0\left(m^*-m\right)\ ,\ M+\lambda_0\left(M^*-M\right)\right), \\ &where:(m,M)\in\gamma_{c_0},\ \left(m^*,M^*\right)\in\gamma_{c_0^*}, \\ &\left((1-\lambda)m+\lambda m^*,(1-\lambda)M+\lambda M^*\right)\subset D_{c_0^*}\setminus D_{c_0}\ \ for\ \ 0<\lambda<1,\ \ |c_0^*-c_0|=|F'(\lambda_0)|\Big\}.\end{aligned}$$

In other words, the set γ consists of the points (m',M') such that:

- $(m',M')\in\mu$;
- $\mu=\left\{\left(m_\mu,M_\mu\right);\ m_\mu=(1-\lambda)m+\lambda m^*,\ M_\mu=(1-\lambda)M+\lambda M^*,\ 0\le\lambda\le 1\right\}$;
- $(m,M)\in\gamma_{c_0}$, $\left(m^*,M^*\right)\in\gamma_{c_0^*}$;
- $\mu\subset\overline{D_{c_0^*}\setminus D_{c_0}}$;
- $|c_0^*-c_0|=\left|\left\langle gradU(m',M'),\left(m^*-m,\ M^*-M\right)\right\rangle\right|$.

It is clear that γ is a connected set and for every points $(m,M)\in\gamma_{c_0}$ and $\left(m^*,M^*\right)\in\gamma_{c_0^*}$ the segment μ intersects the set γ. According to Theorem 2, it is possible to chose two points $\left(m^*,M^*\right)\in\gamma_{c_0^*}$ and $(m',M')=\left(m+\lambda_0\left(m^*-m\right)\ ,\ M+\lambda_0\left(M^*-M\right)\right)\in\gamma\cap\mu$ so that the vector

$$\left(m^* - m\ ,\ M^* - M\right)$$

to be collinear with the vector

$$gradU\left(m', M'\right) = gradU\left(m + \lambda_0\left(m^* - m\right)\ ,\ M + \lambda_0\left(M^* - M\right)\right).$$

Then from (4) it follows

$$\left|c_0^* - c_0\right| = \left\langle\ gradU\left(m + \lambda_0\left(m^* - m\right)\ ,\ M + \lambda_0\left(M^* - M\right)\right),\right.$$

$$\left.\frac{gradU\left(m + \lambda_0\left(m^* - m\right)\ ,\ M + \lambda_0\left(M^* - M\right)\right)}{\left\|gradU\left(m + \lambda_0\left(m^* - m\right)\ ,\ M + \lambda_0\left(M^* - M\right)\right)\right\|}\right\rangle . \sqrt{\left(m^* - m\right)^2 + \left(M^* - M\right)^2}$$

$$= \left\|gradU\left(m + \lambda_0\left(m^* - m\right)\ ,\ M + \lambda_0\left(M^* - M\right)\right)\right\| . \rho_E\left(\left(m, M\right), \left(m^*, M^*\right)\right).$$

Therefore, we have

$$\rho_E\left(\left(m, M\right), \left(m^*, M^*\right)\right) = \frac{\left|c_0^* - c_0\right|}{\left\|gradU\left(m + \lambda_0\left(m^* - m\right), M + \lambda_0\left(M^* - M\right)\right)\right\|},$$

whence we obtain

$$\frac{\left|c_0^* - c_0\right|}{\sup\left\{\left\|gradU\left(m', M'\right)\right\|, \left(m', M'\right) \in D\right\}}$$
$$= const_1 \le \rho_E\left(\left(m, M\right), \left(m^*, M^*\right)\right)$$
$$\le const_2 = \frac{\left|c_0^* - c_0\right|}{\inf\left\{\left\|gradU\left(m', M'\right)\right\|, \left(m', M'\right) \in D\right\}}.$$

From the last inequalities, we get:

(5) $$const_1 \ \le\ \inf\left\{\inf\left\{\rho_E\left(\left(m^*, M^*\right), \left(m, M\right)\right), \left(m^*, M^*\right) \in \gamma_{c_0^*}\right\}, \left(m, M\right) \in \gamma_{c_0}\right\}$$
$$= \rho_E\left(\gamma_{c_0^*}, \gamma_{c_0}\right)$$

and

(6) $$\sup\left\{\inf\left\{\rho_E\left(\left(m^*, M^*\right), \left(m, M\right)\right), \left(m^*, M^*\right) \in \gamma_{c_0^*}\right\}, \left(m, M\right) \in \gamma_{c_0}\right\} \le const_2 .$$

By analogy with (6) we conclude that

(7) $$\sup\left\{\inf\left\{\rho_E\left(\left(m^*, M^*\right), \left(m, M\right)\right), \left(m, M\right) \in \gamma_{c_0}\right\}, \left(m^*, M^*\right) \in \gamma_{c_0^*}\right\} \le const_2 .$$

Using (6) and (7) we have

(8) $$\rho_H\left(\gamma_{c_0^*},\gamma_{c_0}\right)=\max\left\{\sup\left\{\inf\left\{\rho_E\left(\left(m^*,M^*\right),\left(m,M\right)\right),\left(m^*,M^*\right)\in\gamma_{c_0^*}\right\},\left(m,M\right)\in\gamma_{c_0}\right\},\right.$$
$$\left.\sup\left\{\inf\left\{\rho_E\left(\left(m^*,M^*\right),\left(m,M\right)\right),\left(m,M\right)\in\gamma_{c_0}\right\},\left(m^*,M^*\right)\in\gamma_{c_0^*}\right\}\right\}\le const_2.$$

Therefore, from (5) and (8) it is follows

$$const_1\le\rho_E\left(\gamma_{c_0^*},\gamma_{c_0}\right)\le\rho_H\left(\gamma_{c_0^*},\gamma_{c_0}\right)\le const_2$$

and hence

$$\frac{\rho_H\left(\gamma_{c_0^*},\gamma_{c_0}\right)}{\rho_E\left(\gamma_{c_0^*},\gamma_{c_0}\right)}\le\frac{const_2}{const_1}=\frac{\sup\left\{\left\|gradU\left(m',M'\right)\right\|,\left(m',M'\right)\in D\right\}}{\inf\left\{\left\|gradU\left(m',M'\right)\right\|,\left(m',M'\right)\in D\right\}}=\kappa\ .$$

The theorem is proved.

As corollary of the last theorem we obtain the following theorem.

Theorem 4. *The solution of problem (1), (2), (3) is orbital Hausdorff stable.*

Proof. Let ε be arbitrary positive constant, $c_0=U\left(m_0,M_0\right)$ and the constants c_1 and c_2 satisfy the inequalities $0<c_1<c_0<c_2$. For example let $c_1=\frac{1}{2}c_0$ and $c_2=\frac{3}{2}c_0$. Then, according to Theorem 3, it follows that the system (1), (2) is orbital gravitating in the domain $D=D_{c_2}\setminus D_{c_1}$ with a constant $\kappa=\kappa\left(D\right)=\kappa\left(c_1,c_2\right)=\kappa\left(c_0\right)$ $=\kappa\left(m_0,M_0\right)$. Then for every point $\left(m_0^*,M_0^*\right)\in D$ such that

$$\rho_E\left(\left(m_0^*,M_0^*\right),\left(m_0,M_0\right)\right)<\varepsilon/\kappa$$

we have

$$\rho_H\left(\gamma_{c_0^*},\gamma_{c_0}\right)\le\kappa\rho_E\left(\gamma_{c_0^*},\gamma_{c_0}\right)\le\kappa\rho_E\left(\left(m_0^*,M_0^*\right),\left(m_0,M_0\right)\right)<\varepsilon,$$

where $c_0^*=U\left(m_0^*,M_0^*\right)$.

The theorem is proved.

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