Structural Identification of Quasi White Noise Linear Models

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Abstract

This paper shows that within the parametric admissible region of the Box and Jenkins ARMA(p,q) structures there is a sub-region where the classical identification procedure via ACF and PACF does not work properly. We define this sub-space the "Quasi White Noise region" and show how the bootstrap procedure can be used to improve the performance of the structural identification for series located in this particular region. A simulation study was carried out and a comparison between the traditional and the bootstrap procedures is presented.

Keywords: Box and Jenkins, Bootstrap, Quasi White Noise Process

1. *Introduction*

The Box and Jenkins approach, proposed in 1970, [1], as a general procedure to develop time series models for forecasting and control, consists of three stages, namely: structural identification, model parameter estimation and goodness of fitting tests. The structural identification is carried out via joint use of the autocorrelation function (ACF for short) and the partial autocorrelation function (PACF for short). However, in this particular stage of the procedure, there is a chances of the one assumes that the time series in study has been generated by a white noise type of stochastic process, when in fact, this assumption is not true. In other words the straight forward application of the identification procedure leads one to admit the series as being a white noise, yet, the model parameters are located in a particular region of the parametric space with properties not defected by the classical procedure. We define a time series with this property as "Quasi White Noise" series (QWN) from now on.

The bootstrap is a non-parametric computational intensive statistical procedure (CIS for short), proposed in 1979, [2] and [3], which allows the evaluation of the variability of any statistics based on information of a single sample. It is recommended for situations where the standard procedures are not available or are difficult to be obtained in analytical terms. This technique can be used in problems characterized by both; finite sample or big sample size, as they produce better results in the former and as good result as the usual asymptotic in the latter (*). In this paper we formally propose the use of bootstrap as an additional technique to be used in the model identification stage of Box and Jenkins ARIMA structures. The remainder of the paper is organized as follows. In section 2 we address the problems associated with the identification of time series showing the QWN property, while in section 3 we propose the use of bootstrap for QWN series. In section 4 we describe a simulation study carried out and the main conclusions of the proposed procedure are drawn. The paper is concluded with an appendix containing the results and plots from the simulation study.

2. Classical Identification of Arima Structures and the Qwn Series

It is well known that the identification of the ARIMA structure of the process generating a given series is carried out via the joint use of the (1-α)% confidence intervals for the ACF (ρ_k) and the PACF (Φ_{kk}) given by (see Box and Jenkins [1] for details:

$$I = [-z_{(1-\alpha/2)}.\sqrt{\hat{V}(\hat{\rho}_k)}\ ;\ z_{(1-\alpha/2)}.\sqrt{\hat{V}(\hat{\rho}_k)}\ ] \quad \text{for } \rho_{k,} \text{ and}$$

$$II = [-z_{(1-\alpha/2)}.\ \frac{1}{\sqrt{n}}\ ;\ z_{(1-\alpha/2)}.\ \frac{1}{\sqrt{n}}\ ] \quad \text{for } \phi_{kk}$$

As described in Box and Jenkins the intervals above are used to test, respectively, the following hypothesis: H_{01}: $\rho_k = 0$ and H_{02}: $\Phi_{kk} = 0$. Therefore, the joint use of both: the sample correlogram (graph ρ_k x k) and the sample partial correlogram (graph Φ_{kk} x k) and the application of the tests for every k = 1, 2, ... results in the identification of an ARIMA structure candidate to model the series under study. However, situations may occur whereby the sample ACF and PACF are different of zero; however their values are not big enough to reject the null hypothesis of either one or both tests. In this particular case, the straightforward use of the classical procedure would mislead us in accepting the series as being generated by a white noise process. In reality, we are facing models whose parameters are located in particular regions of the parametric space whose theoretical absolute values of ρ_k and ϕ_{kk} are small. Models presenting this property are the object of this paper and are formally defined below. Let T = {1, 2, 3, ... , n} represent a finite set of natural numbers and I and II above the (1-α)% confidence interval to test, respectively: $\rho_k = 0$ and $\phi_{kk} = 0$. We define the QWN process as:

Def.: The QWN is as an ergotic stochastic process whose absolute values of ρ_k and ϕ_{kk}, though different of zero, are located within intervals I and II respectively.

The regions of the stationary parametric space of the following structures ARMA(p,q) , i.e.: AR(1), AR(2), MA(1), MA(2) and ARMA(1,1) where the QWN processes exist are marked below, in figures I to V through the sets Q and Q^2 within each one of the admissible (stationary and invertible) regions. The corresponding proofs of the existence of such regions can be found in reference [4]. Then denoting c = f(α,n) where f(.) is a non-negative function of the confidence level α and of the series size n, we have, for each structure:

AR(1

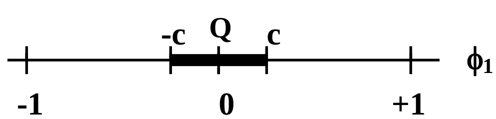

Figure I - AR(1) : Q = { $\phi_1 \in \Re$; | ϕ_1 | < c }

AR(2)

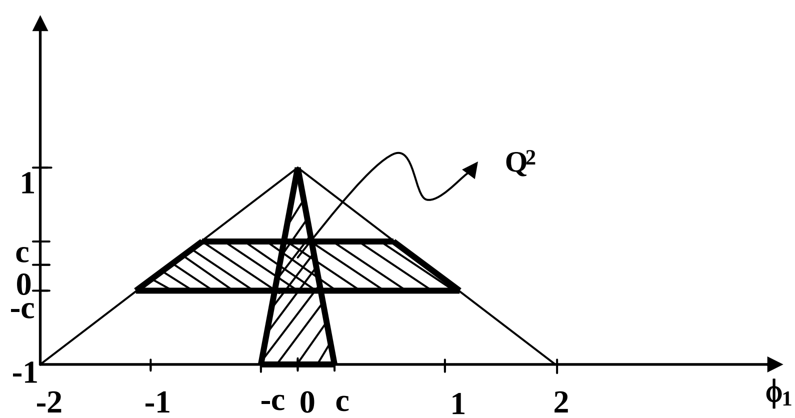

Figure II - AR(2) : $Q^2 = \{(\phi_1, \phi_2) \in \Re^2 ; |\phi_1| < c(1 - \phi_2) \text{ and } |\phi_2| < c\}$

MA(1)

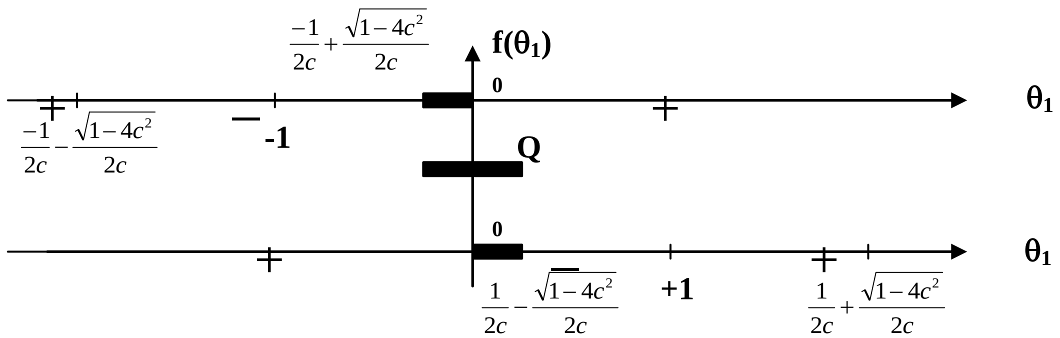

Figure III - MA(1) : Q = { $\theta_1 \in \Re$ **;** $|\theta_1| < \frac{1}{2c} - \frac{\sqrt{1-4c^2}}{2c}$ **}**

MA(2)

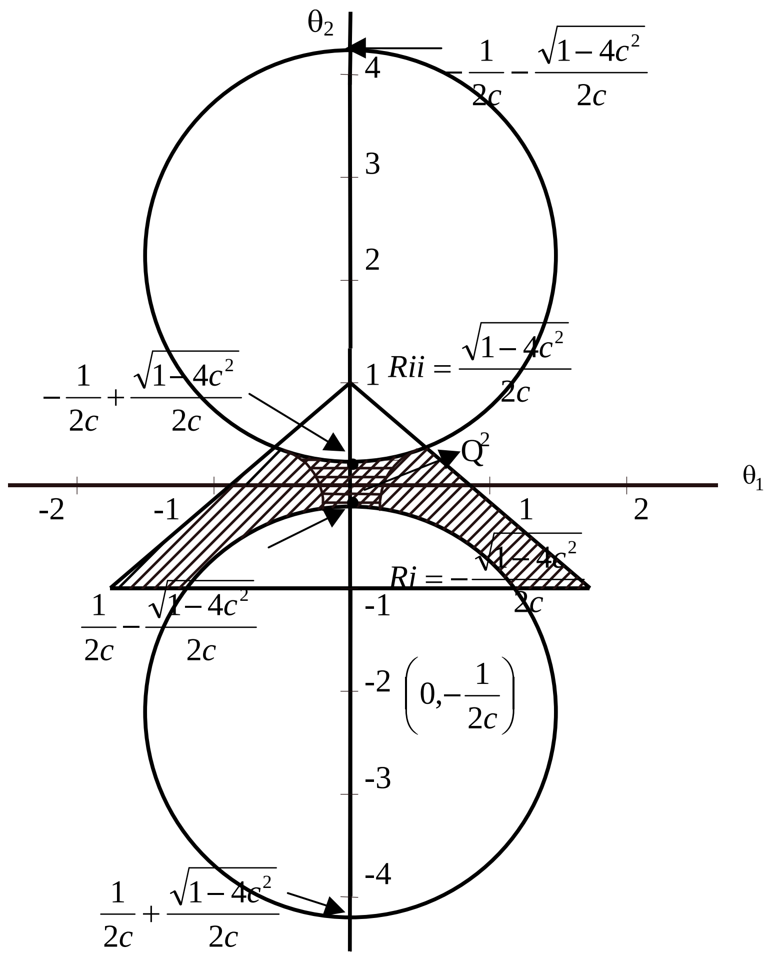

Figure IV - MA(2): $Q^2 = \{(\theta_1, \theta_2) \in \Re^2$ **;** $c\theta_1^2 + c\theta_2^2 + \theta_1 - \theta_1\theta_2 + c > 0$**,** $c\theta_1^2 + c\theta_2^2 - \theta_1 + \theta_1\theta_2 + c > 0$**,** $\theta_1^2 + \theta_2^2 + \frac{1}{c}\theta_2 + 1 > 0$ **and** $\theta_1^2 + \theta_2^2 - \frac{1}{c}\theta_2 + 1 > 0\}$

ARMA(1,1)

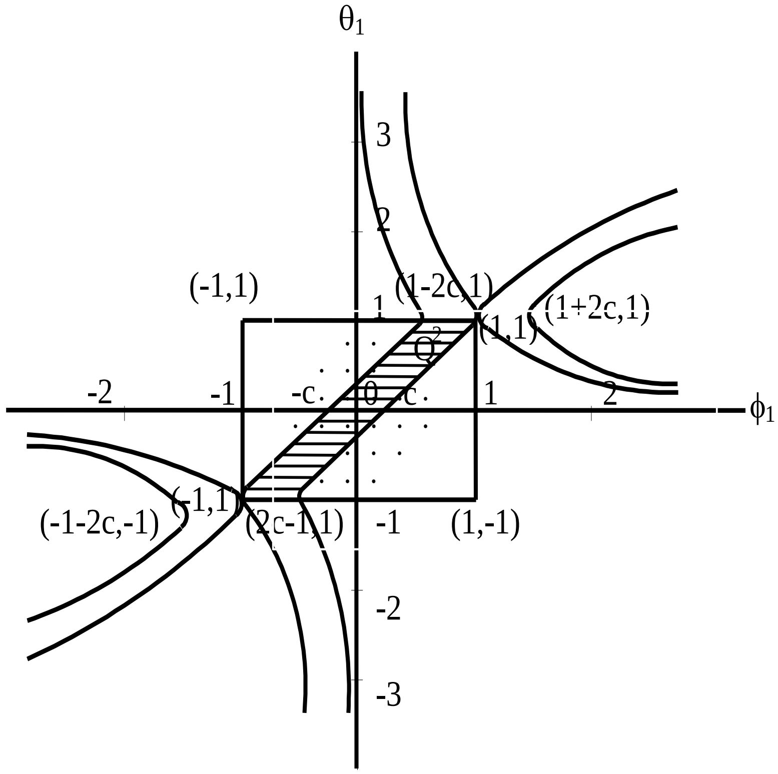

Figure V - ARMA(1,1) : $Q^2 = \{(\phi_1, \theta_1) \in \Re^2$; $(\phi_1 - c)\theta_1^2 - (\phi_1^2 - 2c\phi_1 + 1)\,\phi_1 + \phi_1 - c < 0$, $(\phi_1 + c)\theta_1^2 - (\phi_1^2 + 2c\phi_1 + 1)\theta_1 + \phi_1 + c > 0$ \}

3. Using the Bootstrap in Structural Identification

As mentioned before the bootstrap can be very useful in helping the identification of ARMA(p,q) structures, [6] and [5]. Models with parameters within of the QWN region, in particular, can be better identified by the use of the bootstrap. In order to use this technique in the structural identification of process generating the series under study one must first calculate the bootstrap's distributions of $\hat{\rho}_k$ and $\hat{\Phi}_{kk}$. The algorithm used to generate such distributions is as shown in the flowchart of figure VI. Once these distributions are obtained, one can use them to estimate probability intervals for the true unknown parameters. The bootstrap interval for the parameters (ρ_k and Φ_{kk}) used in this study is the "Bias Corrected Percentile Interval" as described in [3] and has the following expression, $[C\hat{D}F^{-1}(\Phi(2z_0 - z_\alpha)); C\hat{D}F^{-1}(\Phi(2z_0 + z_\alpha))]$, which is used for either one ρ_k and Φ_{kk}, depending on the bootstrap distribution considered.

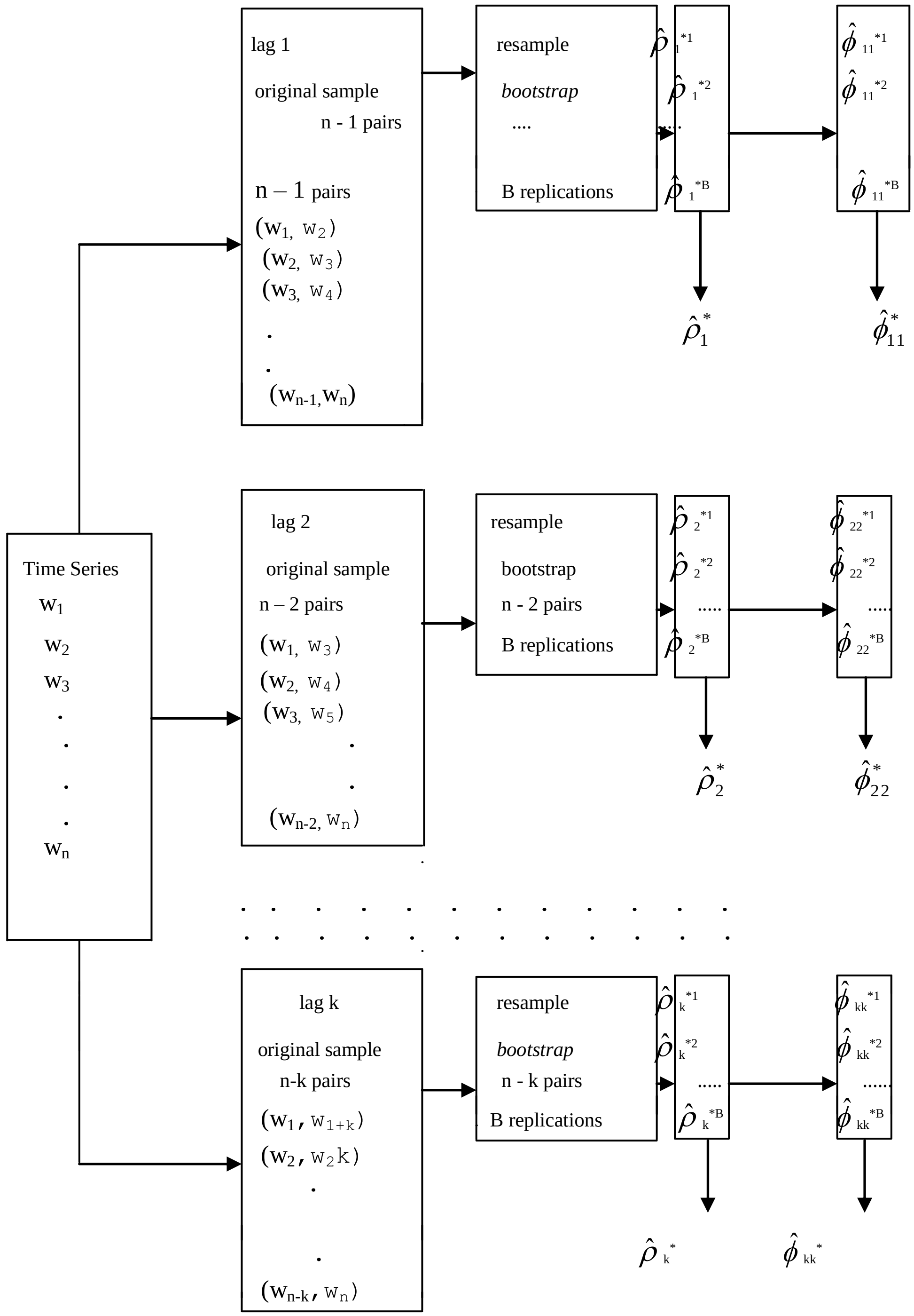

Figure VI - Flowchart of Algorithm Used to Generate the Bootstrap Distributions of $\hat{\rho}_k$ AND $\hat{\phi}_{kk}$

4. Study of Simulation and Conclusions

We artificially generated time series corresponding to models within the five ARMA(p,q) structures considered in this study namely: AR(1), AR(2), MA(1), MA(2) and ARMA(1,1). The positions of the parametersΦ’s and θ’s within of the respective parametric spaces of each one of the five structures are shown in figures VII to XI. The number of models for each structure was: 14 AR(1); 12 AR(2); 14 MA(1); 19 MA(2) and 24 ARMA(1,1). The residual variance was fixed at σ_a^2 = 0.1 for all series and a Gaussian distribution with mean zero for the noise was assumed throughout [4]. In each experiment were worked 100 Monte Carlo replications and 1000 bootstrap replications.

In the evaluation of the procedure classical versus the bootstrap procedure was calculated the empirical power of the hypothesis tests concerning the ACF and PACF. In the classical procedure the following null hypothesis were considered:

H_{01}: $\rho_k = 0$ and H_{02}: $\phi_{kk} = 0$

and was checked whether intervals I and II include the zero value. Concerning the bootstrap procedure the null hypothesis were the following:

H_{01}:[$\hat{\rho}_k^{*\ell o}$, $\hat{\rho}_k^{*up}$] $\supset 0$ and H_{02}:[$\hat{\phi}_{kk}^{*\ell o}$, $\hat{\phi}_{kk}^{*up}$] $\supset$

Where the superscripts $*\ell o$ and $*up$ denote the bounds upper and lower for the corresponding parameters, and, the above intervals are the BC percentile intervals defined previously.

Was, also, used the Euclidean metric to evaluate the distance between the number of coverage of the true parameter by the classical intervals and the number of expected coverage for each simulated model [7]. As were considered the first 3 lags (k =1, 2, 3), then they compose jointly the number of coverage as a single point. Likewise the expected coverage jointly was also considered a single point and the "distance" between these two points was evaluated by the Euclidean metric. In a similar way, was also evaluated the "distance" between the single point corresponding to the joint coverage for the three first lags with the bootstrap intervals and a second single point corresponding to the joint expected coverage for the first three lags. Incidentally, this second single point, formed via expected coverage is the same for both: classical and bootstrap procedures. The graphs these distances and the locations of the parameters of models in the parameter space for each one of the ARMA(p, q) structures simulated are shown in the figures VII to XXI of appendix I. The estimated standard errors of ρ_k and ϕ_{kk} for the first three lags are as shown in appendix II. From the results of the simulation study we can draw the following conclusions:

(I) The BC percentile interval has a greater power of rejecting the null hypothesis of zero values for the ACF and PACF (when they are false), especially when the models have their parameters within the QWN region. For models with parameters near to the boundaries of the stationary and invertibility regions, this power decreases, particularly for the PACF. This can be easily seen in the corresponding plots of the Euclidean distances in appendix I, where the distances of the bootstrap points to the corresponding expected ones are, in general, smaller than the classical distances.

(II) The bootstrap estimates of $\hat{\rho}_k$ and $\hat{\Phi}_{kk}$ are better for models located in the QWN regions (or near them), as can be seen by the estimates of these quantities shown in tables 1 to 5 of appendix II.

(III) For time series generated by real white noise process, the performance of the bootstrap method is superior to the classical procedure and is according to the estimation of the standard error, as shown in table 6 of the appendix II.

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Appendix I

Location of model's parameters in the parameter space and Euclidean distance between the bootstrap (B) and the expected value and between the classical (C) and the expected value for structure's models.

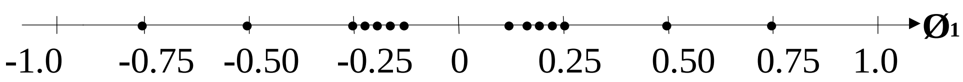
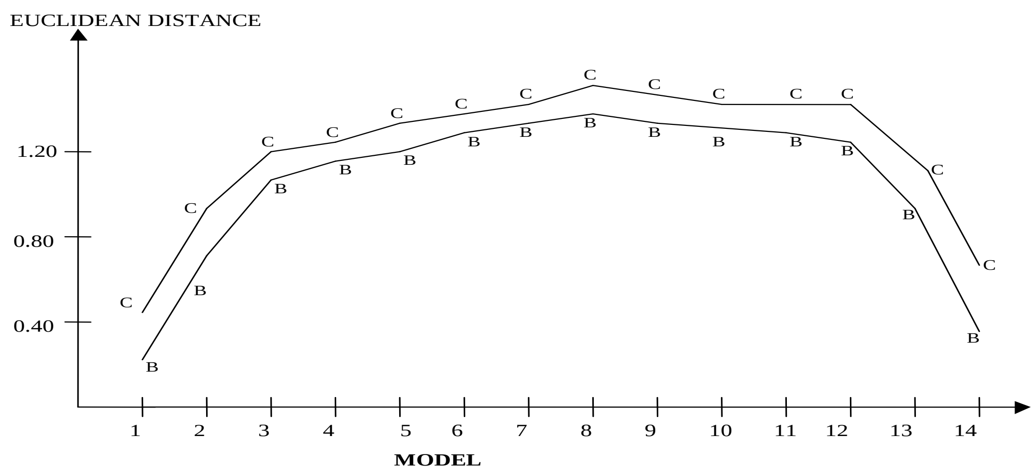


Figure VII - Location of Model's Parameters in the Parameter Space for the AR (1) Structure



FigureVIII - Euclidean Distance between the Bootstrap (B) and the Expected value and between the Classical (C) and the Expected Value for Models of the AR (1) Structure for ACF

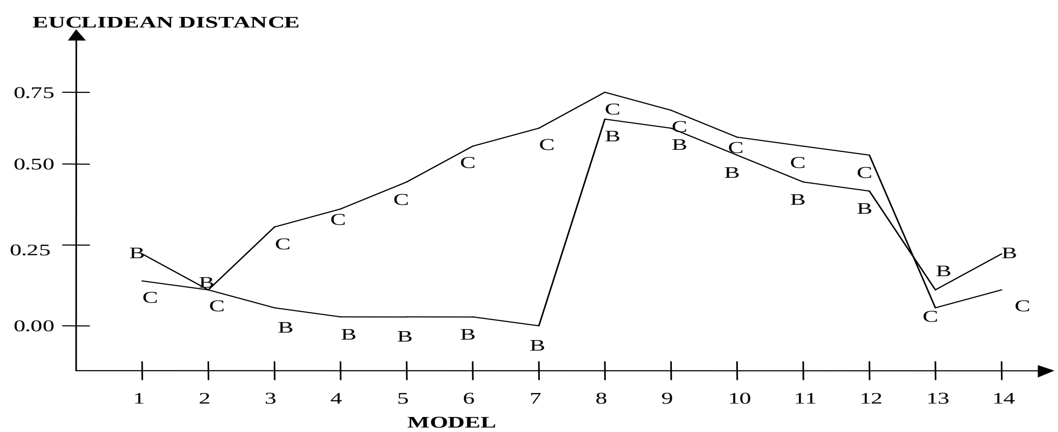


Figure IX - Euclidean Distance between the Bootstrap (B) and the Expected Value and between the Classical (C) and the Expected Value for Models of the AR (1) Structure for PACF

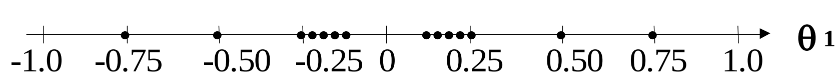


Figure X - Location of Model's Parameters in the Parameter Space for the MA (1) structure

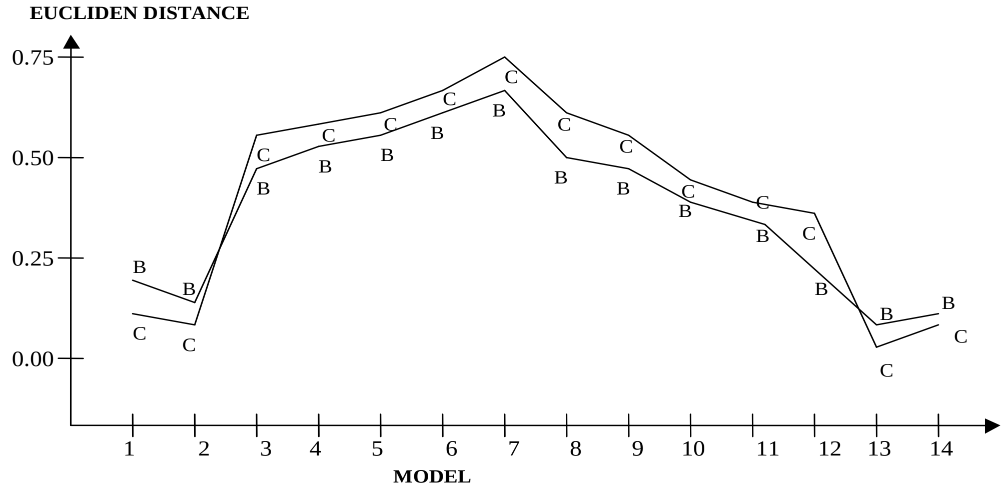


Figure XI – Euclidean Distance between the Bootstrap (B) and the Expected Value and between the Classical (C) and the Expected Value for Models of the MA(1) Structure for ACF

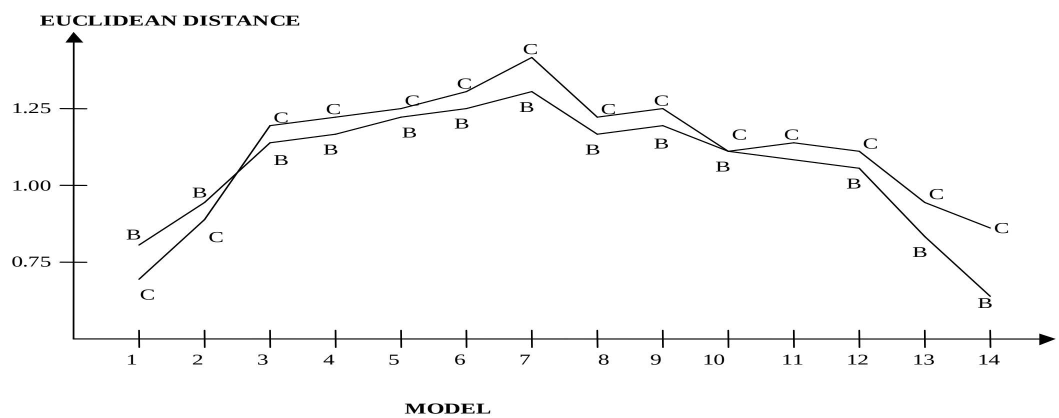


Figure XII - Euclidean Distance between the Bootstrap (B) and the Expected Value and between the Classical (C) and the Expected Value for Models of the MA (1) Structure for PACF

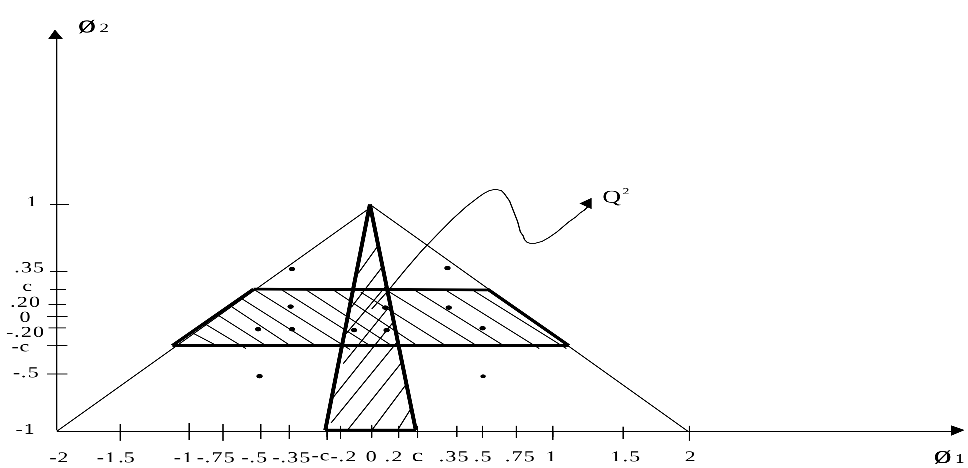


Figure XIII - Location of Model's Parameters in the Parameter Space for the AR (2) Structure

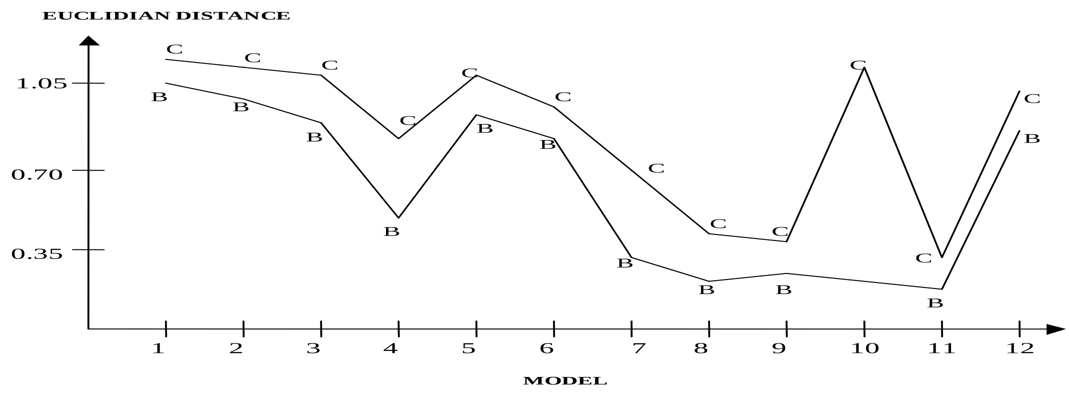


Figure XIV- Euclidean Distance between the Bootstrap (B) and the Expected Value and between the Classical (C) and the Expected Value for Models of the AR (2) Structure for ACF

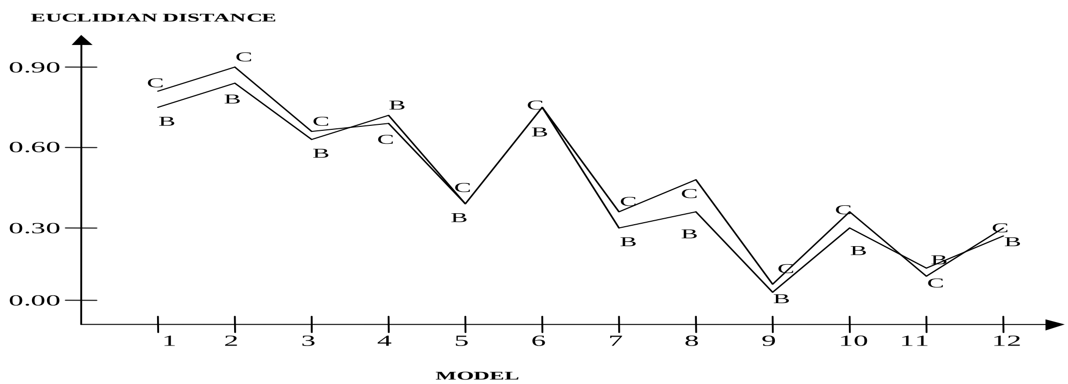


Figure XV - Euclidean Distance between the Bootstrap (B) and the Expected Value and Between the Classical (C) and the Expected Value for Models of the AR (2) Structure for PACF

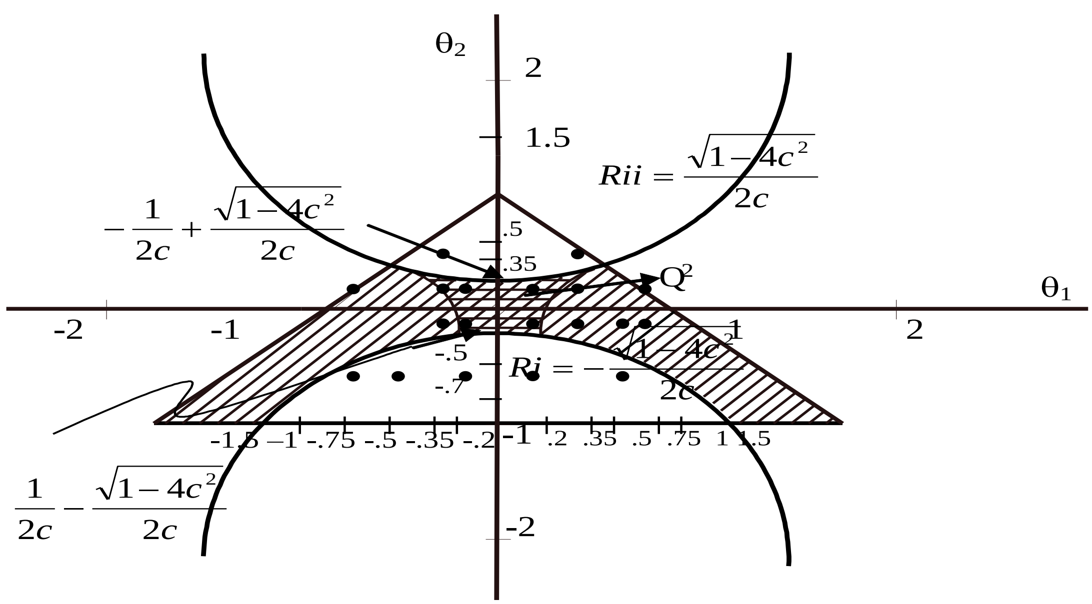


Figure XVI - Location of Model’s Parameters in the Parameter Space for the MA (2) Structure

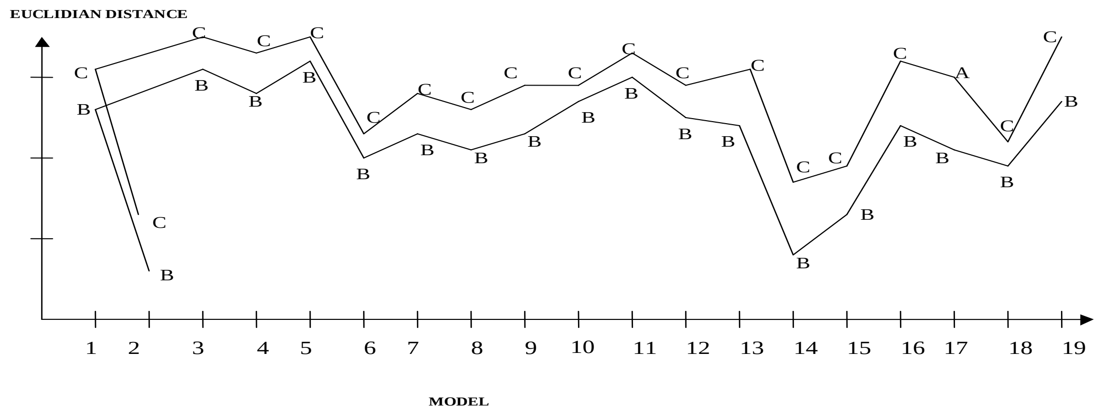


Figure XVII: Euclidean Distance between the Bootstrap (B) and the Expected Value and between the Classical (C) and the Expected Value for Models of the MA(2) Structure for ACF

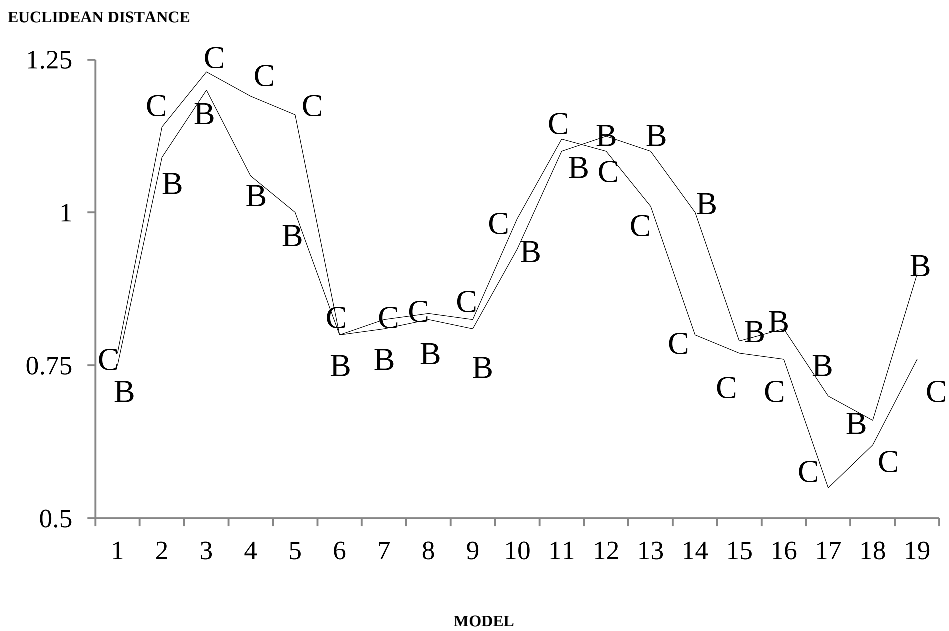


Figure XVIII - Euclidean Distance between the Bootstrap (B) and the Expected Value and between the Classical (C) and the Expected value for Models of the MA (2) Structure for PACF

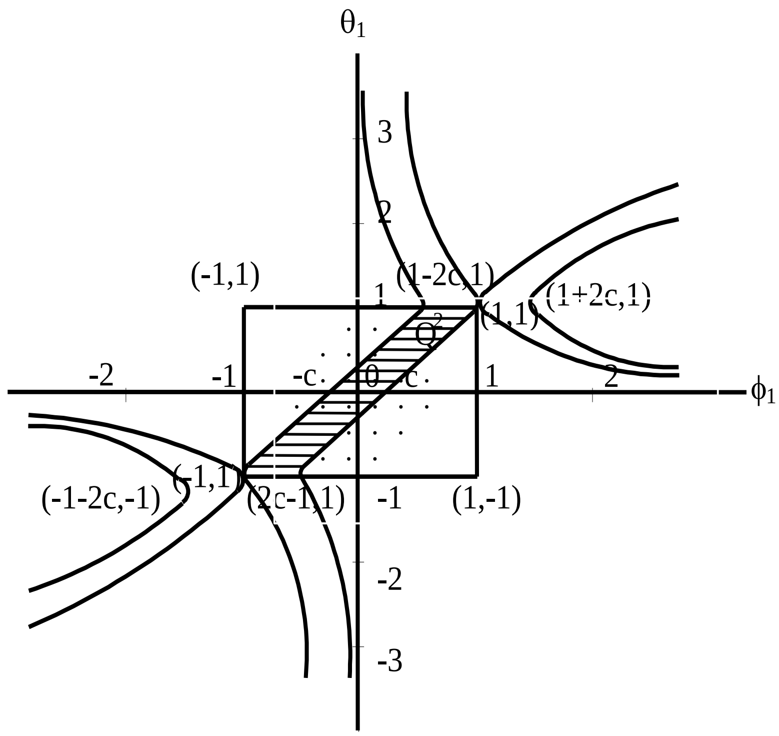


Figure XIX - Location of Model's Parameters in the Parameter Space for the ARMA (1,1) Structure

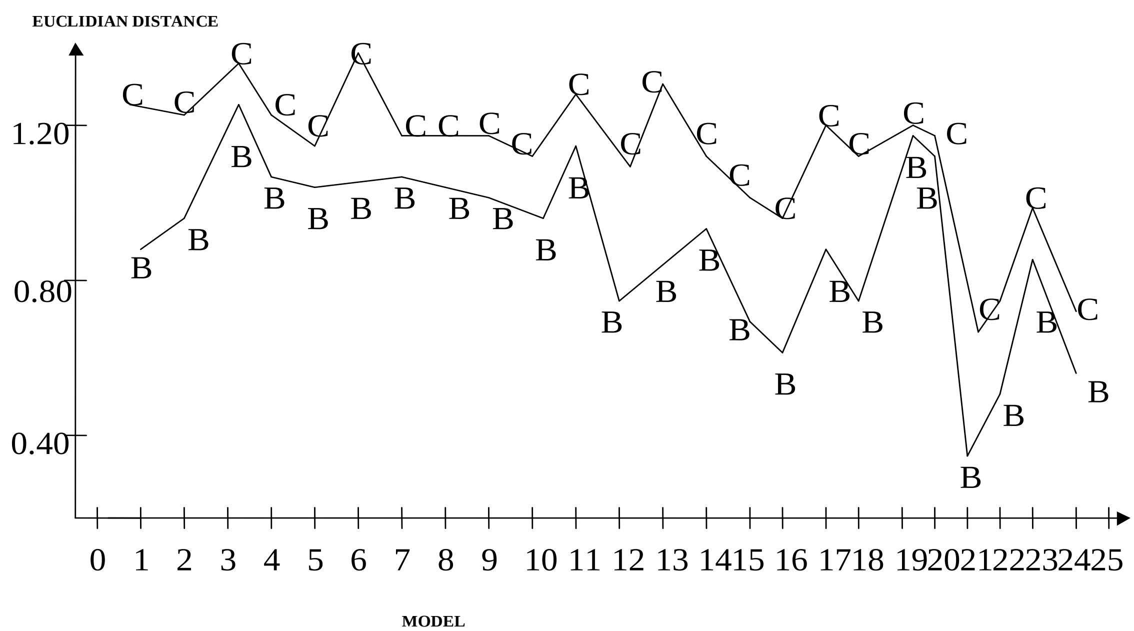


Figure XX - Euclidean Distance between the Bootstrap (B) and the Expected Value and between the Classical (C) and the Expected value for Models of the ARMA (1, 1) Structure for ACF

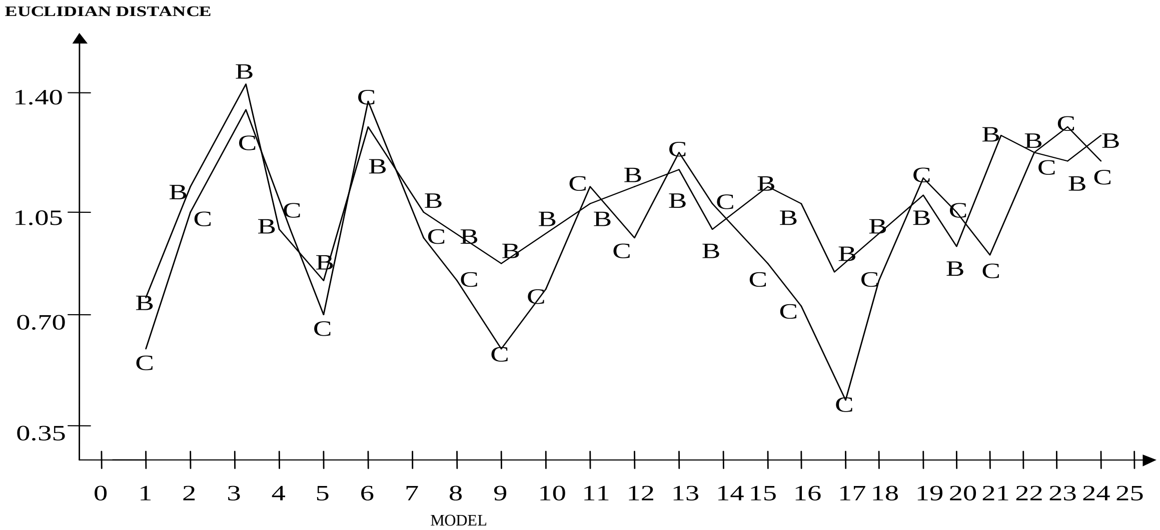


Figure XXI – Euclidean Distance between the Bootstrap (B) and the Expected Value and between the Classical (C) and the Expected value for Models of the ARMA(1,1) Structure for PACF

<u>Appendix II</u>: Standard errors estimates of the classical and bootstrap procedures for $\hat{\rho}_k$ and $\hat{\Phi}_{kk}$, lags k = 1, 2, 3

Tab. 1 – Models of the AR (1) Structure

Model – n Parameters	$\sigma(\hat{\rho}_k)$	$s(\hat{\rho}_k)$	$s(\hat{\rho}^*_k)$	$\sigma(\hat{\Phi}_{kk})$	$s(\hat{\Phi}_{kk})$	$s(\hat{\Phi}^*_{kk})$
AR(1) n=50	.136	.141	.135	.136	.141	.135
Φ_1=0.125	.140	.145	.133	.134	.141	.137
	.135	.148	.131	.139	.141	.138
AR(1) n=50	.136	.141	.135	.136	.141	.135
Φ_1=0.150	.141	.145	.133	.134	.141	.137
	.135	.148	.131	.139	.141	.138
AR(1) n=50	.135	.141	.136	.135	.141	.136
Φ_1=0.200	.142	.147	.133	.134	.141	.139
	.135	.150	.130	.139	.141	.139
AR(1) n=50	.135	.141	.137	.135	.141	.137
Φ_1=0.225	.143	.148	.133	.133	.141	.140
	.136	.151	.130	.139	.141	.140
AR(1) n=50	.134	.141	.137	.134	.141	.137
Φ_1=0.250	.144	.149	.133	.133	.141	.141
	.137	.152	.130	.139	.141	.141
AR(1) n=50	.123	.141	.146	.123	.141	.146
Φ_1=0.500	.153	.166	.132	.134	.141	.167
	.151	.171	.126	.137	.141	.163
AR(1) n=50	.102	.141	.153	.102	.141	.153
Φ_1=0.750	.151	.193	.136	.135	.141	.255
	.172	.211	.125	.136	.141	.239
AR(1) n=50	.133	.141	.135	.133	.141	.135
Φ_1=-.125	.142	.147	.134	.139	.141	.140
	.139	.149	.132	.138	.141	.141
AR(1) n=50	.133	.141	.135	.133	.141	.135
Φ_1=-.150	.143	.148	.134	.140	.141	.141
	.140	.150	.132	.138	.141	.142
AR(1) n=50	.131	.141	.136	.131	.141	.136
Φ_1=-.200	.145	.150	.134	.141	.141	.143
	.142	.152	.132	.137	.141	.145
AR(1) n=50	.130	.141	.136	.130	.141	.136
Φ_1=-.225	.146	.151	.134	.142	.141	.145
	.143	.154	.132	.136	.141	.146
AR(1) n=50	.129	.141	.137	.129	.141	.137
Φ_1=-.250	.147	.152	.134	.142	.141	.147
	.144	.155	.132	.136	.141	.148
AR(1) n=50	.115	.141	.146	.115	.141	.146
Φ_1=-.500	.159	.172	.134	.147	.141	.182
	.162	.179	.131	.131	.141	.181
AR(1) n=50	.098	.141	.156	.098	.141	.156
Φ_1=-.750	.161	.201	.139	.146	.141	.299
	.181	.224	.131	.124	.141	.288

Tab. 2 – Models of the MA (1) Structure

Model – n Parameters	$\sigma(\hat{\rho}_k)$	$s(\hat{\rho}_k)$	$s(\hat{\rho}^*_k)$	$\sigma(\hat{\Phi}_{kk})$	$s(\hat{\Phi}_{kk})$	$s(\hat{\Phi}^*_{kk})$
MA(1)n=50	.132	.141	.135	.132	.141	.135
θ_1=0.125	.142	.146	.134	.139	.141	.140
	.139	.149	.132	.138	.141	.141
MA(1)n=50	.130	.141	.135	.130	.141	.135
θ_1=0.150	.142	.147	.134	.139	.141	.141
	.140	.150	.132	.137	.141	.142
MA(1)n=50	.127	.141	.136	.127	.141	.136
θ_1=0.200	.144	.149	.134	.140	.141	.143
	.142	.152	.132	.136	.141	.145
MA(1)n=50	.125	.141	.136	.125	.141	.136
θ_1=0.225	.145	.150	.134	.140	.141	.144
	.143	.153	.132	.135	.141	.146
MA(1)n=50	.124	.141	.137	.124	.141	.137
θ_1=0.250	.146	.151	.134	.140	.141	.146
	.144	.154	.132	.135	.141	.148
MA(1)n=50	.106	.141	.142	.106	.141	.142
θ_1=0.500	.155	.162	.134	.134	.141	.162
	.155	.165	.132	.126	.141	.171
MA(1)n=50	.097	.141	.146	.097	.141	.146
θ_1=0.750	.162	.170	.134	.124	.141	.174
	.161	.173	.132	.113	.141	.194
MA(1)n=50	.135	.141	.135	.135	.141	.135
θ_1=-.125	.140	.145	.133	.134	.141	.137
	.135	.148	.131	.139	.141	.138
MA(1)n=50	.134	.141	.135	.134	.141	.135
θ_1=-.150	.140	.145	.133	.134	.141	.137
	.135	.148	.131	.139	.141	.138
MA(1)n=50	.132	.141	.136	.132	.141	.136
θ_1=-.200	.142	.147	.133	.133	.141	.139
	.136	.150	.130	.138	.141	.140
MA(1)n=50	.130	.141	.137	.130	.141	.137
θ_1=-.225	.142	.147	.133	.132	.141	.140
	.136	.151	.130	.137	.141	.141
MA(1)n=50	.129	.141	.137	.129	.141	.137
θ_1=-.250	.143	.148	.133	.132	.141	.141
	.137	.151	.130	.137	.141	.142
MA(1)n=50	.114	.141	.144	.114	.141	.144
θ_1=-.500	.152	.159	.132	.124	.141	.153
	.144	.162	.128	.126	.141	.161
MA(1)n=50	.106	.141	.149	.106	.141	.159
θ_1=-.750	.158	.166	.131	.116	.141	.164
	.149	.170	.127	.111	.141	.183

Tab. 3 – Models of the AR (2) Structure

Model – n Parameters	$\sigma(\hat{\rho}_k)$	$s(\hat{\rho}_k)$	$s(\hat{\rho}^*_k)$	$\sigma(\hat{\Phi}_{kk})$	$s(\hat{\Phi}_{kk})$	$s(\hat{\Phi}^*_{kk})$
AR(2)n=50	.115	.141	.136	.115	.141	.136
Φ_1= 0.200	.136	.146	.136	.128	.141	.140
Φ_2= -0.200	.138	.153	.131	.131	.141	.147
AR(2)n=50	.158	.141	.136	.158	.141	.136
Φ_1= 0.200	.142	.149	.134	.134	.141	.143
Φ_2= 0.200	.141	.155	.129	.146	.141	.142
AR(2)n=50	.111	.141	.135	.111	.141	.135
Φ_1= -0.200	.138	.147	.137	.134	.141	.143
Φ_2= -0.200	.145	.154	.133	.133	.141	.151
AR(2)n=50	.142	.141	.142	.142	.141	.142
Φ_1= -0.350	.156	.167	.135	.146	.141	.173
Φ_2= 0.200	.159	.179	.131	.135	.141	.173
AR(2)n=50	.107	.141	.139	.107	.141	.139
Φ_1= -0.350	.144	.154	.135	.139	.141	.150
Φ_2= -0.200	.150	.159	.134	.132	.141	.161
AR(2)n=50	.151	.141	.141	.151	.141	.141
Φ_1= 0.350	.147	.159	.135	.132	.141	.159
Φ_2= 0.200	.151	.169	.127	.146	.141	.155
AR(2)n=50	.169	.141	.139	.169	.141	.139
Φ_1= 0.350	.144	.166	.138	.128	.141	.175
Φ_2= 0.350	.167	.185	.127	.149	.141	.177
AR(2)n=50	.155	.141	.144	.155	.141	.144
Φ_1= -0.350	.156	.176	.139	.144	.141	.202
Φ_2= 0.350	.170	.200	.132	.133	.141	.209
AR(2)n=50	.076	.141	.141	.076	.141	.141
Φ_1= -0.500	.129	.157	.140	.125	.141	.155
Φ_2= -0.500	.153	.171	.138	.128	.141	.216
AR(2)n=50	.101	.141	.143	.101	.141	.143
Φ_1= -0.500	.152	.164	.134	.142	.141	.165
Φ_2= -0.200	.157	.167	.133	.129	.141	.177
AR(2)n=50	.081	.141	.142	.081	.141	.142
Φ_1= 0.500	.127	.155	.139	.119	.141	.152
Φ_2= -0.500	.136	.171	.137	.117	.141	.210
AR(2)n=50	.107	.141	.144	.107	.141	.144
Φ_1= 0.500	.147	.160	.131	.131	.141	.155
Φ_2= -0.200	.144	.163	.130	.128	.141	.166

Tab. 4 – Models of the MA (2) Structure

Model – n Parameters	$\sigma(\hat{\rho}_k)$	$s(\hat{\rho}_k)$	$s(\hat{\rho}^*_k)$	$\sigma(\hat{\Phi}_{kk})$	$s(\hat{\Phi}_{kk})$	$s(\hat{\Phi}*_{kk})$
MA(2)n=50	.145	.141	.136	.145	.141	.136
θ_1= 0.200	.141	.152	.134	.145	.141	.146
θ_2= -0.200	.144	.157	.131	.139	.141	.147
MA(2)n=50	.172	.141	.137	.172	.141	.137
θ_1= 0.200	.122	.154	.138	.138	.141	.154
θ_2= -0.500	.152	.169	.130	.151	.141	.158
MA(2)n=50	.150	.141	.136	.150	.141	.136
θ_1= -0.200	.138	.148	.133	.135	.141	.140
θ_2= -0.200	.137	.152	.129	.143	.141	.140
MA(2)n=50	.113	.141	.135	.113	.141	.135
θ_1= 0.200	.134	.146	.137	.127	.141	.143
θ_2= 0.200	.145	.154	.132	.129	.141	.149
MA(2)n=50	.117	.141	.135	.117	.141	.135
θ_1= -0.200	.132	.145	.137	.122	.141	.140
θ_2= 0.200	.141	.153	.131	.131	.141	.146
MA(2)n=50	.177	.141	.135	.177	.141	.135
θ_1= -0.200	.117	.149	.137	.126	.141	.146
θ_2= -0.500	.146	.163	.128	.154	.141	.150
MA(2)n=50	.109	.141	.137	.109	.141	.137
θ_1= 0.350	.140	.151	.137	.126	.141	.147
θ_2= 0.200	.149	.157	.132	.124	.141	.157
MA(2)n=50	.105	.141	.136	.105	.141	.136
θ_1= 0.350	.128	.147	.140	.117	.141	.146
θ_2= 0.350	.152	.159	.132	.119	.141	.159
MA(2)n=50	.109	.141	.137	.109	.141	.137
θ_1= -0.350	.140	.151	.137	.126	.141	.147
θ_2= 0.200	.149	.157	.132	.124	.141	.157
MA(2)n=50	.112	.141	.137	.112	.141	.137
θ_1= -0.350	.126	.146	.139	.109	.141	.144
θ_2= 0.350	.147	.160	.130	.122	.141	.157
MA(2)n=50	.133	.141	.141	.133	.141	.141
θ_1= -0.350	.144	.155	.132	.140	.141	.149
θ_2= -0.200	.142	159	.128	.138	.141	.148
MA(2)n=50	.126	.141	.140	.126	.141	.140
θ_1= 0.350	.148	.161	.134	.152	.141	.159
θ_2= -0.200	.152	.165	.132	.135	.141	.160
MA(2)n=50	.108	.141	.145	.108	.141	.145
θ_1= 0.500	.155	.169	.134	.151	.141	.174
θ_2= -0.200	.160	.173	.132	.132	.141	.178
MA(2)n=50	.120	.141	.146	.120	.141	.146
θ_1= 0.500	.143	.174	.137	.178	.141	.190
θ_2= -0.500	.170	.185	.130	.127	.141	.186

Tab. 4 (cont.) – *MA (2)*

MA(2)n=50	.127	.141	.146	.127	.141	.147
θ_1= -0.500	.136	.167	.134	.163	.141	.171
θ_2= -0.500	.159	.176	.126	.128	.141	.166
MA(2)n=50	.091	.141	.150	.091	.141	.150
θ_1= 0.750	.165	.179	.134	.132	.141	.195
θ_2= -0.200	.169	.182	.132	.123	.141	.215
MA(2)n=50	.102	.141	.142	.102	.141	.142
θ_1= 0.750	.152	.160	.136	.120	.141	.159
θ_2= 0.200	.155	.165	.132	.112	.141	.175
MA(2)n=50	.112	.141	.145	.112	.141	.145
θ_1= -0.750	.149	.157	.133	.112	.141	.152
θ_2= 0.200	.145	.164	.129	.133	.141	.169
MA(2)n=50	.096	.141	.153	.096	.141	.153
θ_1= -0.750	.160	.174	.130	.126	.141	.179
θ_2= -0.200	.156	.177	.126	.121	.141	.197

Tab. 5 - *ARMA(1,1)*

Model–n-parameters	$\sigma(\hat{\rho}_k)$	$s(\hat{\rho}_k)$	$s(\hat{\rho}^*{}_k)$	$\sigma(\hat{\Phi}_{kk})$	$s(\hat{\Phi}_{kk})$	$s(\hat{\Phi}*_{kk})$
ARMA(1,1) n=50	.096	.141	.155	.096	.141	.155
Φ_1=.200	.164	.177	.130	.116	.141	.185
θ_1=-.750	.158	.180	.126	.106	.141	.212
ARMA(1,1) n=50	.119	.141	.143	.119	.141	.143
Φ_1=.200	.151	.159	.131	.131	.141	.153
θ_1=-.250	.144	.162	.128	.133	.141	.156
ARMA(1,1) n=50	.134	.141	.134	.134	.141	.134
Φ_1=.200	.140	.145	.134	.137	.141	.137
θ_1= .250	.137	.148	.132	.138	.141	.138
ARMA(1,1) n=50	.117	.141	.137	.117	.141	.137
Φ_1=.200	.146	.152	.135	.135	.141	.147
θ_1=.500	.146	.156	.132	.131	.141	.152
ARMA(1,1) n=50	.106	.141	.141	.106	.141	.141
Φ_1=.200	.153	.160	.135	.126	.141	.158
θ_1= .750	.154	.163	.132	.118	.141	.170
ARMA(1,1) n=50	.135	.141	.134	.135	.141	.134
Φ_1= -.200	.139	.144	.134	.135	.141	.136
θ_1= -.250	.135	.147	.131	.139	.141	.137
ARMA(1,1) n=50	.112	.141	.142	.112	.141	.142
Φ_1= -.200	.155	.163	.134	.142	.141	.163
θ_1= -.250	.155	.167	.132	.131	.141	.168
ARMA(1,1) n=50	.096	.141	.148	.096	.141	.148
Φ_1= -.200	.163	.175	.134	.132	.141	.186
θ_1= .500	.165	.179	.132	.121	.141	.201
ARMA(1,1) n=50	.090	.141	.151	.090	.141	.151
Φ_1= -.200	.168	.181	.134	.121	.141	.201
θ_1= .750	.170	.185	.131	.109	.141	.229
ARMA(1,1) n=50	.115	.141	.144	.115	.141	.144
Φ_1= -.200	.150	.157	.133	.116	.141	.151
θ_1= -.750	.144	.162	.129	.120	.141	.162

ARMA(1,1) n=50	.123	.141	.138	.123	.141	.138
Φ_1= -.200	.143	.149	.133	.126	.141	.142
θ_1= -.500	.138	.153	.130	.133	.141	.146
ARMA(1,1) n=50	.099	.141	.155	.099	.141	.155
Φ_1=.500	.157	.182	.132	.132	.141	.204
θ_1= -.250	.164	.189	.124	.128	.141	.203
ARMA(1,1) n=50	.142	.141	.137	.142	.141	.137
Φ_1=.500	.146	.150	.133	.134	.141	.141
θ_1= .250	.138	.153	.129	.142	.141	.141
ARMA(1,1) n=50	.136	.141	.137	.136	.141	.137
Φ_1= -.500	.150	.154	.134	.143	.141	.148
θ_1= -.250	.145	.158	.132	.137	.141	.149
ARMA(1,1) n=50	.094	.141	.153	.094	.141	.153
Φ_1= -.500	.164	.188	.135	.141	.141	.227
θ_1= .250	.175	.198	.130	.126	.141	.231
ARMA(1,1) n=50	.082	.141	.157	.082	.141	.157
Φ_1= -.500	.166	.197	.135	.127	.141	.262
θ_1= .500	.183	.208	.129	.117	.141	.284
ARMA(1,1) n=50	.103	.141	.151	.103	.141	.151
Φ_1= .200	.159	.171	.130	.124	.141	.172
θ_1= -.500	.153	.174	.126	.120	.141	.185
ARMA(1,1) n=50	.084	.141	.160	.084	.141	.160
Φ_1= .500	.158	.192	.132	.123	.141	.235
θ_1= -.500	.171	.200	.123	.116	.141	.251
ARMA(1,1) n=50	.126	.141	.136	.126	.141	.136
Φ_1= -.500	.137	.146	.135	.123	.141	.140
θ_1= -.750	.139	.152	.131	.133	.141	.143
ARMA(1,1) n=50	.121	.141	.136	.121	.141	.136
Φ_1= .500	.140	.148	.135	.131	.141	.143
θ_1= .750	.143	.153	.132	.128	.141	.147
ARMA(1,1) n=50	.078	.141	.158	.078	.141	.158
Φ_1= .750	.145	.206	.136	.131	.141	.322
θ_1= -.250	.177	.227	.124	.128	.141	.313
ARMA(1,1) n=50	.136	.141	.146	.136	.141	.146
Φ_1= .750	.158	.173	.134	.132	.141	.186
θ_1= .250	.162	.185	.127	.143	.141	.178
ARMA(1,1) n=50	.153	.141	.137	.153	.141	.137
Φ_1= .750	.154	.151	.133	.136	.141	.145
θ_1= .500	.146	.156	.129	.145	.141	.143
ARMA(1,1) n=50	.128	.141	.148	.128	.141	.148
Φ_1= -.750	.169	.181	.136	.171	.198	.132
θ_1= -.250	.171	.198	.132	.129	.141	.208

Tab. 6 – WHITE NOISE

White Noise – n parameters	$\sigma(\hat{\rho}_k)$	$s(\hat{\rho}_k)$	$s(\hat{\rho}^*{}_k)$	$\sigma(\hat{\Phi}_{kk})$	$s(\hat{\Phi}_{kk})$	$s(\hat{\Phi}*_{kk})$
n=50	.136	.141	.134	.136	.141	.134
σ_a^2 =.100	.140	.144	.134	.136	.141	.137
	.136	.147	.132	.139	.141	.137