On Vector-valued Littlewood-Paley Theorem

Adam M. Ali
Khartoum Academy for Aviation and Technology
Sudan

Abstract

In this paper we prove the existence of the Banach space –valued Littlewood –Paley theorem implies that a Banach space is isomorphic to a Hilbert space.

Keywords: Vector-Valued, Space-Valued, Hilbert Space, Banach Space

Introduction

Suppose that a function ψ is in $S\left(R^n\right)$ with $\operatorname{supp}\widehat{\psi} \subset \left\{\xi \in R^2 : \frac{1}{2} \le |\xi| \le 2\right\}$ and $\left|\widehat{\psi}\left(\xi\right)\right| \ge c > 0$ if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$. Then one form of the classical Littlewood-Paley theorem on R^n says

$$c\|f\|_p \le \left\| \left\{ \sum_{k \in \mathbb{Z}} |\psi_k * f|^2 \right\}^{\frac{1}{2}} \right\|_p \le C\|f\|_p \tag{1}$$

where $1 < p < \infty, \psi_k\left(x\right) = 2^{kn}\psi\left(2^k x\right)$, and c, C are constants independent of f.

We study the vector-valued Littlewood-Paley theorem. To be precise, let B be a Banach space and $L_B^p\left(R^n\right)$ be the space of strongly measurable B-valued function f for which $|f|_B \in L^p\left(R^n\right)$. It is well known that if B is a Hilbert space, then the classical Littlewood-Paley theorem still holds

$$c\|f\|_{L_B^p} \le \left\| \left\{ \sum_{k \in z} |\psi_k * f|^2 \right\}^{\frac{1}{2}} \right\|_{L_p} \le C\|f\|_{L_B^p} \tag{2}$$

where $1 < p < \infty$ and ψ is the same function as in (1).

We first prove that if B is a Banach space on (2) for one function ψ mentioned above, then (2) holds for a more general family of operators.

Definition (1.1) [3]:

A family of operators $\{S_k\}_{k\in z}$ is said to be an approximation to the identity if for $0 < \varepsilon \le 1$ and $\delta = \varepsilon - \varepsilon' > 0$ there is a constant C such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in R^n$, $S_k\left(x, y\right)$, the kernels of S_k, satisfy the following conditions:

(i) $| S_k\left(x, y\right) \le C \dfrac{2^{-k\varepsilon}}{\left(2^{-k} + |x-y|\right)^{n+\varepsilon}}$

(ii) $\left| S_k(x,y) - S_k(x',y) \ \right| \le C\left(\dfrac{|x-x'|}{2^{-k} + |x-y|}\right)^{\varepsilon} \left| \dfrac{2^{-k\varepsilon}}{\left(2^{-k} + |x-y|\right)^{n+\varepsilon}} \right.$ for $|x-x'| \le \frac{1}{2}\left(2^{-k} + |x-y|\right)$,

(iii) $|S_k(x,y) - S_k(x,y')| \le C\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon} \frac{2^{-k\varepsilon}}{\left(2^{-k}+|x-y|\right)^{n+\varepsilon}}$ for $|y-y'| \le \frac{1}{2}\left(2^{-k}+|x-y|\right)$

(iv) $\left|\left[S_k(x,y)-S_k(x,y')\right]-\left[S_k(x',y)-S_k(x',y')\right]\right|$

$$\le C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon'}\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon'} \frac{2^{-k\delta}}{\left(2^{-k}+|x-y|\right)^{n+\delta}} \text{ for } |x-x'| \le \frac{1}{2}\left(2^{-k}+|x-k|\right)$$

and $|y-y'| \le \frac{1}{2}\left(2^{-k}+|x-y|\right)$, and $\delta = \varepsilon - \varepsilon' > 0$

(v) $\int S_k(x,y)\,dy = \int S_k(x,y)\,dx = 1$ for all $k \in \mathbb{Z}$.

All of the conditions (i) – (v) on the approximate identities are needed for the Calderon reproducing formula.

Definition (1.2) [3]:

Fix two exponents $0 < \beta \le 1$ and $\gamma > 0$. A B–valued function f, where B is a Banach space, is said to be a test function of type (β,γ) centered at $x_0 \in R^n$ with width $d > 0$ if f satisfies the following conditions:

(i) $|f(x)|_B \le C\frac{d\gamma}{\left(d+|x-x_0|^{n+\gamma}\right)}$,

(ii) $|f(x)-f(x')|_B \le C\left(\frac{|x-x'|}{d+|x-x_0|}\right)^{B}\frac{d^{\gamma}}{\left(d+|x-x_0|\right)^{n+\gamma}}$ for $|x-x'| \ge \frac{1}{2}\left(d+|x-x_0|\right)$,

(iii) $\int_{R^n} f(x)\,dx = 0$

The collection of all test functions of type (β,γ) centered at with width $d > 0$ will be denoted by $M_B^{(\beta,\gamma)}(x_0,d)$. If $f \in M_B^{(\beta,\gamma)}(x_0,d)$ the norm of f in $f \in M_B^{(\beta,\gamma)}(x_0,d)$ is defined by

$$\|f\|_{M_B^{(\beta,\gamma)}(x_0,d)} = \inf\{C \ge 0\}$$

If (i),(ii),(iii) of Definition (1.2) hold, we denote the class of all $f \in M_B^{(\beta,\gamma)}(0,1)$ by $M_B^{(\beta,\gamma)}$. It is easy to see that $M_B^{(\beta,\gamma)}$ is a Banach space under the norm $f \in M_B^{(\beta,\gamma)} < \infty$. It is also easy to see that $M_B^{(\beta,\gamma)} = f \in M_B^{(\beta,\gamma)}(x_0,d)$ for $x_0 \in R^n$ and $d > 0$, with equivalent norms.

Theorem (1.3) [3]:

Suppose that $\{S_k\}$ is approximation to the identity defined in (4) below. Set $D_k = S_k - S_{k-1}$. Then there exists a family of operators $\{\tilde{D}_k\}_{k\in\mathbb{Z}}$ such that for all $f \in M_B^{(\beta,\gamma)}$,

$$f = \sum_{k\in\mathbb{Z}} \tilde{D}_k D_k(f) \tag{3}$$

where the series converges in the norm of $M_B^{(\beta,\gamma)}$ with $\beta' < \beta$ and $\gamma' < \gamma$. Moreover, $\tilde{D}_k(x,y)$ the kernel of $\tilde{D}_k$, satisfy the following estimates: for ε', $0 < \varepsilon' < \varepsilon$ where ε is the regularity exponent of S_k, there exists a constant $C > 0$.

Such that

(i) $\left|\tilde{D}_k(x,y)\right| \le C \dfrac{2^{-k\varepsilon'}}{\left(2^{-k}+|x-y|\right)^{n+\varepsilon'}}$

(ii) $\left|\tilde{D}_k(x,y)-\tilde{D}_k(x',y)\right| \le C\left(\dfrac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon'} \dfrac{2^{-k\varepsilon'}}{\left(2^{-k}+|x-y|\right)^{n+\varepsilon'}}$ for $|x-x'| \le \dfrac{1}{2}\left(2^{-k}+|x-y|\right)$

(iii) $\int \tilde{D}_k(x,y)dy = \int \tilde{D}_k(x,y)dx = 0$ for all $k \in Z$.

Since $\sum_{k\in z} D_k(f) = f$ in the strong topology of $L_B^2(R^n)$, it is easy to see that $M_B^{(\beta,\gamma)}$ is dense in $L_B^2(R^n)$ for all $0<\beta\le 1$ and $\gamma>0$.

Theorem (1.4) [3]:

Suppose that B is a Banach space. Suppose $\{S_k\}$ is an approximation to the identity and $D_k = S_k - S_{k-1}$, and the Littlewood-Paley theorem holds for $\{D_k\}$, that is, for $1<p<\infty$,

$$c\|f\|_{L_B^p} \le \left\| \left\{ \sum_{k\in\mathbb{Z}} \left|D_k(f)\right|^2 \right\}^{\frac{1}{2}} \right\|_{L_B^p} \le C\|f\|_{L_B^p} \qquad (4)$$

Then the Littlewood-Paley theorem holds for $\{E_k\}$ where $E_k = R_k - R_{k-1}$ and $\{R_k\}$ is an approximation to the identity, that is, for $1<p<\infty$

$$c\|f\|_{L_B^p} \le \left\| \left\{ \sum_{k\in\mathbb{Z}} \left|E_k(f)\right|^2 \right\}^{\frac{1}{2}} \right\|_{L_B^p} \le C\|f\|_{L_B^p} \qquad (5)$$

Proof:

We need to show (5) for all $f \in M_B^{(\beta,\gamma)}$. Suppose that (5) holds and $E_k = R_k - R_{k-1}$ where $\{R_k\}$ is an approximation to the identity. By Theorem (1.3) for all $f \in M_B^{(\beta,\gamma)}$ with $0<\beta\le 1$ and $\gamma>0$, we have

$$E_k(f) = \sum_{j\in z} E_k \tilde{D}_j D_j(f)$$

It is easy to check that $E_k\tilde{D}_j(x,y)$, the kernel of $E_k\tilde{D}_j$, satisfies the following estimates

$$\left|E_k\tilde{D}_j(x,y)\right| \le C 2^{-|k-j|\varepsilon''} \frac{2^{-(k\wedge j)\varepsilon''}}{\left(2^{-(k\wedge j)}+|x-y|\right)^{n+\varepsilon''}} \qquad (6)$$

where $0<\varepsilon''<\varepsilon'<\varepsilon$ and $a\wedge b$ denotes the minimum of a and b

Hence

$$\left\| \left\{ \sum_{k\in Z} \left|E_k(f)\right|^2 \right\}^{\frac{1}{2}} \right\|_p \le C \left\| \left\{ \sum_{k\in Z} \left[\sum_{j\in Z} 2^{-|k-j|\varepsilon''} \left|M\left(D_j(f)\right)\right| \right]^2 \right\}^{\frac{1}{2}} \right\|_p$$

$$\le C \left\| \left\{ \sum_{j\in Z} \left|M\left(D_j(f)\right)\right|^2 \right\}^{\frac{1}{2}} \right\|_p$$

$$\leq C\left\|\left\{\sum_{j\in Z}\left|D_j(f)\right|^2\right\}^{\frac{1}{2}}\right\|_p \tag{7}$$

Where M is the Hardy-Littlewood maximal function; the last inequality follows from the Fefferman-Stein vector-valued maximal inequality [1].

The proof of the inverse inequality of (7) is the same, and hence, this completes the proof.

Corollary (1.5)[2]: Consider the assumption of Theorem (1.4) ,then

$$C\sum_{j=1}^{\infty}\|f_j\|_{L_B^p} \leq \left\|\left\{\sum_{j=1}^{\infty}\sum_{k\in Z}\left|D_k(f_j)\right|^2\right\}^{1/2}\right\|_{L_B^p} \leq C\sum_{j=1}^{\infty}\|f_j\|_{L_B^p},$$

and

$$C\sum_{j=1}^{\infty}\|f_j\|_{L_B^p} \leq \left\|\left\{\sum_{j=1}^{\infty}\sum_{K\in Z}^{\infty}\left|E_K(f_j)\right|^2\right\}^{1/2}\right\| \leq C\sum_{j=1}^{\infty}\|f_j\|_{L_\beta^p},$$

for all $f_j \epsilon M_B(\beta, r)$.

Proof: for $0 < \beta < 1$ and $r > 0$ we have

$$\sum_{j=1}^{\infty}E_K(f_j) = \sum_{j=1}^{\infty}\sum_{i\in R}E_K\widetilde{D}_iD_i(f_j)$$

$E_K\widetilde{D}_i(x.y)$ is the kernel of $E_K\widetilde{D}_i$ by Theorem (14). Satisfying the estimates of (6) in Theorem (1.4).Then we have

$$\left\|\left\{\sum_{K\in Z}\sum_{j=1}^{\infty}\left|E_K(f_j)\right|^2\right\}^{\frac{1}{2}}\right\|_p \leq C\left\|\left(\sum_{K\in Z}\left[\sum_{i\in Z}\sum_{j=1}^{\infty}2^{-|K-i|\varepsilon''}\left|M\left(D_i(f_j)\right)\right|\right]^2\right)^{\frac{1}{2}}\right\|_p \leq C\left\|\left\{\sum_{i\in Z}\sum_{j=1}^{\infty}\left|M\left(D_i(f_i)\right)\right|^2\right\}^p\right\|_p^{\frac{1}{2}}$$

$$\leq C\left\|\left\{\sum_{i\in Z}\sum_{j=1}^{\infty}\left|D_j(f_j)\right|^2\right\}^{\frac{1}{2}}\right\|_p$$

Theorem (1.6) [3]:

Let B be any Banach space, $n\geq 1$, $a_0\in B,...,a_n\in B$. Let $1<\lambda_1<...<\lambda_n<\lambda_{n+1},...,\lambda_j$ be the integers for all j and

$$2\pi\left\{\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1+\lambda_2}{\lambda_3}+...+\frac{\lambda_1+\lambda_2+...\lambda_n+...}{\lambda_{n+1}}\right\}\leq\alpha<1 \tag{8}$$

Let $F_n(\theta_1,...,\theta_n)=a_0+a_1e^{i\theta_1}+...+a_ne^{i\theta_n}$, $\quad 0\leq\theta_k\leq 2\pi$ for $1\leq k\leq n$ and

$F_n(t)=a_0+a_1e^{i\lambda_1 t}+...+a_ne^{i\lambda_n t}$, $\quad 0\leq\theta_k\leq 2\pi$. Then,

$$(1-\alpha)\|F_n\|_2\leq\|F_n\|_2\leq(1+\alpha)\|F_n\|_2 \tag{9}$$

where , the L^2 –norm will always be the normalized $L^2[0,2\pi]^n$ norm or $L^2[0,2\pi]$ with respect to all the variables.

To prove this theorem, set

$$F_{n,k}(t,\theta_{k+1},...\theta_n)=a_0+a_1e^{i\lambda_1 t}+...+a_ke^{i\lambda_k t}+a_{k+1}e^{i\theta_{k+1}}+...+a_ne^{i\theta_n} \tag{10}$$

We will prove

$$\|F_{n,k-1}\|_2-\varepsilon_k\|f_n\|_2\leq\|f_{n,k}\|_2\leq\|f_{n,k-1}\|_2+\varepsilon_k\|f_n\|_2 \tag{11}$$

where $1\leq k\leq n$ and $\varepsilon_k=2\pi\dfrac{\lambda_1+...+\lambda_{k-1}}{\lambda_k}$.

Observe first that once (11) is proved we obviously obtain

$$\left(1-\varepsilon_1-\ldots-\varepsilon_n\right)\left\|F_n\right\|_2 \le \left\|F_{n,k}\right\|_2 \le \left\|F_{n,k-1}\right\|_2 + \varepsilon_k \left\|F_n\right\|_2 \le \left(1+\varepsilon_1+\ldots+\varepsilon_n\right)\left\|F_n\right\|_2 \qquad (12)$$

which yields the theorem.

The second observation is that the inequality in (11) for $1 \le k < n$ follows from the inequality in (11) for $k = n$. Indeed, let us freeze $\theta_{k+1},\ldots,\theta_n$ an write

$$\widehat{a}_0 = a_0 + a_{k+1}e^{i\theta_{k+1}} + \ldots + a_n e^{i\theta_n}$$

We now apply (11) with n being replaced by k, and obtain

$$\left\|\widehat{a}_0 + a_1 e^{i\lambda_1 t} + \ldots + a_{k-1}e^{i\theta_{k-1}t} + \ldots + a_k e^{i\theta_k}\right\|_{L^2(dtd\theta_k)} - \varepsilon_k \left\|\widehat{a}_0 + a_1 e^{i\theta_1} + \ldots + a_k e^{i\theta_k}\right\|_{L^2(d\theta_1\ldots d\theta_k)}$$

$$\le \left\|\widehat{a}_0 + a_1 e^{i\lambda_1 t} + \ldots + a_k e^{i\lambda_k t}\right\|_{L^2(dtd\theta_k)} + \varepsilon_k \left\|\widehat{a}_0 + a_1 e^{i\theta_1} + \ldots + a_k e^{i\theta_k}\right\|_{L^2(\theta_1\ldots d\theta_k)}$$

Writing ϕ for $\left(\theta_{k+1},\ldots,\theta_n\right)$ and using symbolic notation, we have obtained

$$A(\phi) - \varepsilon_k B(\phi) \le C(\phi) \le A(\phi) + \varepsilon_k B(\phi) \qquad (13)$$

Now we take L^2 norms with respect to ϕ and obtain

$$\left\|A\phi\right\|_{L^2(d\phi)} - \varepsilon_k \left\|B(\phi)\right\|_{L^2(d\phi)} \le \left\|C(\phi)\right\|_{L^2(d\phi)} \qquad (14)$$

$$\le \left\|A(\phi)\right\|_{L^2(d\phi)} + \varepsilon_k \left\|B(\phi)\right\|_{L^2(d\phi)}$$

Here we use the following observation: $f(x) \ge 0$, $g(x) \ge 0, h(x) \ge 0$, and $h(x) \ge f(x) - g(x)$ imply $\|h\|_2 \ge \|f\|_2 - \|g\|_2$, since $h(x) + g(x) \ge f(x)$ obviously implies $\|h\|_2 + \|g\|_2 \ge \|h+g\|_2 \ge \|f\|_2$.

Since $\left\|A(\phi)\right\|_{L^2(d\phi)} = \left\|F_{n,k-1}\right\|_2$, $\left\|B(\phi)\right\|_{L^2(d\phi)} = \left\|F_n\right\|_2$ and $\left\|C(\phi)\right\|_{L^2(d\phi)} = \left\|F_{n,k-1}\right\|_2$, the inequality of (11) with $k = n$. Note first that

$$f^{|\#}{}_n\left(t + \frac{2k\pi}{\lambda_n}\right) = f_{n-1}\left(t + \frac{2k\pi}{\lambda_n}\right) + a_n e^{i\lambda_n t}$$

We now introduce

$$f_n^*(t,k,s) = f_{n-1}\left(t + \frac{2k\pi}{\lambda_n} + \frac{2\pi s}{\lambda_n}\right) + a_n e^{i\lambda_n t}$$

If $0 \le s \le 1$. Then

$$\left|f_n\left(t + \frac{2k\pi}{\lambda_n}\right) - f_n^{\#}(t,k,s)\right|_B \le \left|a_1\right|_B \frac{2\pi\lambda_1}{\lambda_n} + \ldots + \left|a_{n-1}\right|_B \frac{2\pi\lambda_{n-1}}{\lambda_n}$$

$$\le \sup\left(\left|a_1\right|_B,\ldots,\left|a_{n-1}\right|_B\right)\varepsilon_n$$

We obviously have

$$\left|a_k\right|_B \le \left\|F_n\right\|_2 \quad \text{for } 1 \le k \le n$$

Since a_k are the Fourier coefficients of F_n.

Therefore,

$$\left|f_n\left(t + \frac{2k\pi}{\lambda_n}\right) - f_n^{\#}(t,k,s)\right|_B \le \varepsilon_n \left\|F_n\right\|_2 \qquad (15)$$

Taking the L^2 norm with respect to all the variables $t, k \in \left\{0,1,\ldots,\lambda_{n-1}\right\}$ and $S \in [0,1]$, we obtain

$$\left|\left\{\frac{1}{\lambda_n}\sum_{k=0}^{\lambda_{n-1}}\left\|f_n\left(t+\frac{2k\pi}{\lambda_n}\right)\right\|_{L^2(dt)}^2\right\}^{\frac{1}{2}}-\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\frac{1}{\lambda_n}\sum_{k=0}^{\lambda_{n-1}}\left\{\left|f_n^{\#}(t,k,s)\right|dsdt\right\}^{\frac{1}{2}}\right|\leq\varepsilon_n\left\|F_n\right\|_2 \quad (16)$$

But

$$\int_0^1\frac{1}{\lambda_n}\sum_{k=0}^{\lambda_n-1}\left|f_n^{\#}(t,k,s)\right|^2 ds=\frac{1}{\lambda_n}\sum_{k=0}^{\lambda_n-1}\left|\int_0^1 f_{n-1}\left(t+\frac{2k\pi}{\lambda_n}+\frac{2\pi s}{\lambda_n}\right)+a_n e^{i\lambda_n t}\right|^2 ds$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left|f_{n-1}(t+\theta)+a_n e^{i\lambda_n t}\right|^2 d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left|f_{n-1}(\theta)+a_n e^{i\lambda_n t}\right|^2 d\theta \quad (17)$$

Then (16) yields

$$\left|\left\|f_n(t)\right\|_2-\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left|f_{n-1}(\theta)+a_n e^{i\lambda_n t}\right|^2 d\theta dt\right\}^{\frac{1}{2}}\right|\leq\varepsilon_n\left\|f_n\right\|_2 \quad (18)$$

Which is the required estimate

Theorem (1.7) [3]:

Suppose that B is a Banach space. If the B–valued Littlewood-Paley theorem (4) holds for some $1<p_0<\infty$ and where $D_k=S_k-S_{k-1}$ and $\{S_k\}$ is an approximation to the identity, then B is isomorphic to a Hilbert space.

Proof:

First observe that if (4) holds for some $1<p_0<\infty$, then (4) holds for all $1<p<\infty$. We define the operator T on $L_B^{P_0}(R^n)$ by $T(f)=\{D_k(f)\}_{k\in z}$. The fact that (4) holds for means that is a bounded operator from $L_B^{P_0}(R^n)$ to $L_{L_B^2}^{P_0}(R^n)$ where

$$L_{\mathbf{L}_B^2}^{P_0}(R^n)=\left\{(f_k(x))_{k\in z}:\left\|\left\{\sum_{k\in z}\left|f_k(x)\right|_B^2\right\}^{\frac{1}{2}}\right\|_{P_0}<\infty\right\}$$

It is easy to check that T is a vector-valued Calderon-Zygmund operator.

Here we say that an operator T is a vector-valued Calderion. Zygmuand operator if T is a continuous linear operator from $L_B^{P_0}(R^n)$ to $L_{\mathbf{L}_B^2}^{P_0}(R^n)$ for some $1<p_0<\infty$ with the kernel $k(x,y)$ mapping $R^n\times R^n$ to the pace of all bounded operators from B to $\mathbf{L}_B^2$ and satisfy the following conditions: for some $\epsilon>0$, there is a constant $C\geq 0$ such that

$$\left\|k(x,y)\right\|\leq C\left|x-y\right|^n \qquad \forall x,y\in R^n \quad \text{with } x\neq y \quad \text{for all } \left|y-y'\right|\leq\frac{1}{2}\left|x-y\right|, \quad (19)$$

$$\left\|k(x,y)-k(x'-y)\right\|\leq C\left|x-x'\right|^{\varepsilon}\left|x-y\right|^{-(n+\varepsilon)} \text{ for all } \left|x-x'\right|\leq\frac{1}{2}\left|x-y\right|, \quad (20)$$

$$\left\|K(x,y)-K(x',y\right\|\leq C\left|x-x'\right|^{\varepsilon}\left|x-y\right|^{-(n+\varepsilon)}, \text{ for all } \left|x-x'\right|\leq\frac{1}{2}\left|x-y\right|, \quad (21)$$

By the Calderion-Zygmund real-variable theory, T also is bounded from $\mathbf{L}_B^p(R^n)$ to $L_{\mathbf{L}_B^p}^p(R^n)$ for all $1<p<\infty$.

Let $\psi,\phi \in S\left(R^n\right)$ with $\operatorname{supp}\widehat{\psi} \subset \left\{\xi \in R^n : \frac{1}{2} \le |\xi| \le 2\right\}$, $\left|\widehat{\psi}\left(\xi\right)\right| \ge c > 0$ if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$, and $\operatorname{supp}\widehat{\phi} \subset \left\{\xi \in R^n : |\xi| \le 1\right\}$, $\sup_{x\in[0,2\pi]}\left|\phi\left(x\right)\right|^2 \ge \delta > 0$. Suppose that we accept the B –valued Littlewood-Paley theorem in (4) for some $1 < p_0 < \infty$ and $\left\{D_k\right\}$ where $D_k = S_k - S_{k-1}$ and $\left\{S_k\right\}$ is an approximation to the identity. By Theorem (1.4) and the first observation above, we may assume the following inequalities hold:

$$c\|f\|_{L_B^2}^2 \le \sum_{k\in z}\|\psi_k * f\|_{L_B^2}^2 \le C\|f\|_{L_B^2}^2 \qquad (22)$$

where the constants c and C are independent of f.

Now consider the function $f\left(x\right) = f_n\left(x\right)\phi\left(x\right) = \left[a_1 e^{i\lambda_1 x} + ... + a_n e^{i\lambda_n x}\right]\phi\left(x\right)$ where $\lambda_j = 3^{3j}$ for $1 \le j \le n$. Then (22) implies

$$\sum_{j=1}^{n}\left\|a_j\right\|_B^2 \sim \|f_n\|_{L_B^2[0,2\pi]}^2$$

We now apply Theorem (1.6) and obtain

$$\|f_n\|_{L_B^2[0,2\pi]}^2 \sim \left\|a_1 e^{i\theta_1} + ... + a_n e^{i\theta_n}\right\|_{L_B^2[0,2\pi]}^2$$

Now we have

$$\left\|a_1 e^{i\theta_1} + ... + a_n e^{i\theta_n}\right\|_{L_B^2[0,2\pi]}^2 \sim \frac{1}{2^n}\sum_{\varepsilon}\left\|\varepsilon_1 a_1 + ... + \varepsilon_n a_n\right\|_B^2$$

Where the series is extended over all sequences $\varepsilon = \left(\varepsilon_1,...,\varepsilon_k\right)$ with ε_k being independent Bernoulli random variables, that is, $\varepsilon_k = \pm 1$ for $1 \le k \le n$.

This shows that for any $n \ge 1$ and $a_1, a_2, ..., a_n \in B$, there exist constants and such that

$$c\sum_{j=1}^{n}\left\|a_j\right\|_B^2 \le \frac{1}{2^n}\sum\left\|\varepsilon_1 a_1 + ... + \varepsilon_n a_n\right\|_B^2 \le C\sum_{j=1}^{n}\left\|a_j\right\|_B^2$$

which implies that B is isomorphic to a Hilbert space, and hence, Theorem (1.7) is proved.

Results and Discussion

Our main results is collary (1.5) which is a deduction of theorem (1.4) and depends on its main assumption : the set $\left\{S_k\right\}$ is an approximation to the identity .

Conclusion and Recommendations

Does theorem (1.4) holds for $1 \le p \le \infty$?

References

N.N. Vakhania, V.I. Tarieladze, and S.A. Chobanyan, Probability Distributions on Banach Spaces, Mathematics and its Applications (Soviet Series), D. Reidel Publishing Company,1987.

Shawgy Hussein and Adam M.Ali :Plancherel –type and Operator-valued Fourier Multipliers of L^p and Sobolev Spaces, Ph.D. Thesis, Sudan University of Science and Technology (2011).

Y.S.Han and Y. Meyer , A Characterization of Hilbert Spaces and the Vector –valued Littlewood-Paley Theorem, Methods and Applications of Analysis vol. 3-issue (2) 228-234.